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On Almost Expandability in Bitopological Spaces

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Abstract

In this paper, we intend to discuss and investigate some properties that related to the subspaces of the pairwise almost expandable space and some of their characterizations and their relations with some bitopological spaces. Several results are inferred and deduced based on our investigation.

Keywords: *Pairwise almost expandable space; Pairwise normal space; Pairwise regular space; Pairwise paracompact space; Bitopological space.*

1 Introduction

In 1963, Kelly introduced the notion of a bitopological space, i.e. the triple (X, τ_1, τ_2), where X is a non-empty set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular (P-regular), pairwise normal (P-normal) spaces, and obtained generalizations of several standard results such as Urysohn's Lemma and Tietze Extension Theorem. Several authors have since considered the problem of defining compactness for such spaces, see [1, 2, 7, 8]. In particular, Fletcher et. al in [2] gave the definitions of $\tau_1\tau_2$-open and P-open covers in bitopological spaces. Katetove in [3] obtained a necessary and sufficient condition under which every locally-finite collection of closed subsets of a space X can be expanded to a locally-finite collection of open subsets of X. Krajewski in [4] called such a space to be expandable. He proved two main theorems; a Hausdorff developable space is metrizable, and a space X is paracompact if and only if it is θ-refinable and expandable. In fact, Krajewski introduced several generalizations of expandable spaces namely almost-expandable spaces, boundedly expandable spaces, discretely-expandable spaces and others.

Definition 1.1 *A bitopological space $X = (X, \tau_1, \tau_2)$ is said to be τ_1-m-almost expandable with respect to τ_2, where m is an infinite cardinal, if every τ_1-locally finite collection of subsets of X with power less or equal to m is expandable to a τ_2-point finite open collection, i.e. if $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ is a τ_1-locally finite collection of subsets of a τ_1-m-almost expandable space X with $|\Delta| \leq m$, then there exists a τ_2-open, τ_2-point finite collection say $\tilde{G}$ $= \{G_\alpha : \alpha \in \Delta\}$ such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$.*

Definition 1.2 *A bitopological space $X = (X, \tau_1, \tau_2)$ is said to be τ_1-almost expandable space with respect to τ_2, if it is τ_1-m-almost expandable with respect to τ_2 for every infinite cardinal m.*

Definition 1.3 *A bitopological space $X = (X, \tau_1, \tau_2)$ is said to be pairwise almost expandable (P-almost expandable), if it is τ_1-almost expandable with respect to τ_2 and τ_2-almost expandable with respect to τ_1.*

Definition 1.4 *A topological space X is called PF-normal if every point finite open cover of X is normal.*

Definition 1.5 *A bitopological space $X = (X, \tau_1, \tau_2)$ is called pairwise PF-normal ($P - PF$-normal), if every pairwise point finite open cover of X is P-normal.*

Definition 1.6 *A topological space X is called almost discreetly expandable if every discrete collection of subsets of X is expandable to point finite open collection.*

Definition 1.7 *A bitopological space $X = (X, \tau_1, \tau_2)$ is said to be pairwise almost discreatly expandable (P-almost discretely expandable), if every P-discrete collection of subsets of X is P-expandable to a pairwise point finite open collection.*

Next, we intend to state and prove an important result that will definitely help us to deduce some other significant results.

Theorem 1.8 *If a bitopological space $X = (X, \tau_1, \tau_2)$ is P-metacompact, then it is P-almost expandable.*

Proof 1.9 *Let $i \neq j$ in which $i, j = 1, 2$ and let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of subsets of X. Now, for each $x \in X$, let $N(x)$ be a τ_j-open set containing x that intersects only finitely many members of $\tilde{F}$. Define $\tilde{N} = \{N(x) : x \in X\}$. Consequently, since X is P-metacompact, then $\tilde{N}$ has a pairwise point finite open parallel refinement, say $\tilde{V} = \{V_s : V_s is \tau_j - open, S \in \Delta\}$.*

In order to complete our investigation, we need to define $G_\alpha = st(F_\alpha, \tilde{V}) = U\{V \in \tilde{V} : v \cap F_\alpha \neq \phi\}$, for each $\alpha \in \Delta$. Then, $F_\alpha \subseteq G_\alpha$ for $\alpha \in \Delta$, and $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ is τ_j-open finite collection. Therefore, X is a τ_i-almost expandable with respect to τ_j, for $i \neq j$ such that $i, j = 1, 2$. Hence, X is P-almost expandable. Now, based on the previous discussion, we can immediately state and prove the next assertion.

Theorem 1.10 *A P-almost expandable spaces are a strict generalization of P-expandable spaces. This means that every P-expandable space is a P-almost expandable space, but the converse needs not be true.*

Proof 1.11 *First of all, we will begin with proving that the converse of the above assertion needs not be true. For this purpose, we let $X = \mathbb{R}$ and $A = \{1/n : n = 1, 2, 3, \cdots\}$. Define a topology τ on X by letting $u \in \tau$ if $u = O - B$, where $B \subseteq A$ and O is an open set in the usual topology defined on X. Thus; $X = (X, \tau, \tau)$ is P-almost expandable but not P-expandable space. Now, we have the following claim.*
***Claim**: The space X is P-metacompact.*
To prove this claim, we let $i \neq j$ such that $i, j = 1, 2$, and we let $\tilde{U} = \{u_\alpha - B_\alpha : B_\alpha \subseteq A, \alpha \in \Delta\}$ be a τ_i-open cover of X. Thus, the collection $\tilde{U} = \{u_\alpha : \alpha \in \Delta\}$ forms a τ_i-open cover of X. Since (X, τ_u, τ_u) is P-metacompact, then $\tilde{U}_1$ has an open point finite refinement, say $\tilde{V}_1 = \{v_\alpha : v_\alpha$ as τ_j-open $\gamma \in \Gamma\}$. Now, the collection $\{v_\gamma - A : \gamma \in \Gamma\}$ is a τ_j-refinement of $\tilde{U}$ and covers only $X - A$. Since each point of A is contained in some $U_\alpha - B_\alpha$, then we can cover each point $\frac{1}{n}$ of A by some centered open interval I_n of length less than $\frac{1}{2n(n+1)}$ which is contained in some member of $\tilde{U}$ so that these intervals will be disjoint. So, $\{v_\gamma - A : \gamma \in \Gamma\} \cup \{ I_n : n = 1, 2, \cdots\}$ covers X and it is clearly a τ_j-point finite refinement of $\tilde{U}$. Thus, X is P-metacompact. By Theorem 1.8, we can infer that X is P-almost expandable. But $X = (X, \tau, \tau)$ is not P-expandable, because it is not P-countably paracompact. In addition, we can note that if $\tilde{O} = \{O_n : n \in \mathbb{N}\}$, where $O_n = X - (A - \{\frac{1}{n}\})$, then $\tilde{O}$ is a P-countable open cover of X which has no P-open locally finite refinement. This is because in every P-refinement, each P-open set containing zero must intersect infinitely many members of that P-refinement. Therefore, X is not P-countably paracompact.

Lemma 1.12 *A bitopological space X is P-countably metacompact if and only if X is P-almost ω_o-expandable.*

Proof 1.13 *The proof of this result is similar to the proof of Theorem 1.10.*

Definition 1.14 *[5, 6] A bitopological space (X, τ_1, τ_2) is called pairwise countably metacompact (P-countably metacompact), if it is countably τ_i-metacompact and countably τ_2-metacompact.*

Theorem 1.15 *A space $X = (X, \tau_1, \tau_2)$ is P-almost expandable if and only if X is P-almost discreetly expandable and P-countably metacompact.*

Proof 1.16 *Let $i \neq j$ such that $i, j = 1, 2$. If X is P-almost expandable, then it is P-almost discreetly expandable and P-countably metacompact by Lemma 1.12. Conversely, let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of closed subsets of X. For each integer $n \geqslant o$, define $S_n = \{x \in X : ord(x, \tilde{F}) \leq n\}$. Then, $\{ S_n\}_{n=0}^{\infty}$ will be a τ_i-countably open cover of X. Since X is P-countably metacompact, there exist a τ_j-point finite open cover, say $\tilde{V} = \{v_n : n = 0, 1, \cdots\}$, such that $V_n \subseteq S_n$, for each $n \in \mathbb{N}$. Define $H_n = X - \underset{k \succ n}{\cup} V_k$. Then, H_n is τ_j-closed set such that $H_n \subseteq \underset{j=0}{\cup^n} V_j \subseteq S_n$, for each n. We observe that $\{H_n^* = H_n \cap V_n : n = 0, 1, \cdots\}$ is a τ_j-point finite cover of X such that $CL(H_n^*) \subseteq H_n \subseteq S_n$, for each n. If $x \in X$, then choose the largest i such that $x \in V_i$. Thus, $x \notin \underset{k \succ i}{\cup} V_k$ and $x \in H_i^*$. Consequently, $\tilde{F} = \{F(n, a) = CL(H_n^*) \cap F_\alpha : \alpha \in \Delta\}$ is τ_i-bounded locally finite. Since X is P-almost discreetly expandable, there exist, for each n, a τ_j-point finite open collection $\tilde{G}_n = \{G(n, a) : \alpha \in \Delta\}$, such that $F(n, a) \subseteq G(n, a)$, for each $\alpha \in \Delta$. Define $G_n = \underset{\alpha \in \Delta}{\cup} G(n, a)$ so the collection $\tilde{G} = \{G_n : n = 1, 2, \cdots\} \cup \{X - \underset{\alpha \in \Delta}{\cup} F_\alpha\}$ is a τ_j-countable open cover of X. A gain, since X is P-countably metacompact, then $\tilde{G}$ has a τ_i-point finite open refinement, say $\{G_n^* : n = 0, 1, \cdots\}$. Now, consider the collection $\{U_\alpha = \cup_{n=1}^{\infty} \{G(n, a) \cap G_n^*\} : \alpha \in \Delta\}$, which represents a τ_j-point finite open collection such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Hence, X is P-almost expandable.*

The following result aims to show that the P-almost expandable space is equivalent to P-expandable space in the presence of P-PF-normality.

Theorem 1.17 *Let $X = (X, \tau_1, \tau_2)$ be a P-PF-normal bitopological space. Then X is P-expandable if and only if X is P-almost expandable.*

Proof 1.18 *Let $i \neq j$ such that $i, j = 1, 2$, then clearly every expandable is P-almost expandable. Conversely, suppose X is P-almost expandable P-PF-normal space and let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of closed subsets of X. Then, there exists a τ_j-point finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Since $\tilde{G}_1 = \tilde{G} \cup \{X - \underset{\alpha \in \Delta}{\cup} F_\alpha\}$ is a τ_i-point finite open cover of X, then it is normal. Hence, there exists a τ_j-locally finite open cover $\tilde{V} = \{V_s : s \in \Delta\}$ such that $\{st(V_s, \tilde{V}) : s \in \Delta\}$ refines $\tilde{G}_1$. Define $H_\alpha = st(F_\alpha, \tilde{V})$, for each $\alpha \in \Delta$. Then, $\tilde{H} = \{H_\alpha : \alpha \in \Delta\}$ is a τ_j-locally finite open collection such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Therefore, X is P-expandable.*

2 Subspace of pairwise almost expandable space

As we know, not every subspace of a P-almost expandable is P-almost expandable, i.e. the P-almost expandability is not hereditary. But it may be hereditary if we add some conditions to the subspace. This issue is handled by the next results.

Theorem 2.1 *Let $X = (X, \tau_1, \tau_2)$ be a P-almost expandable space and let A be a P-closed subset of X. Then A is P-almost expandable.*

Proof 2.2 *Let A be a P-closed subset of a P-almost expandable space X. Let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of subsets of A. Then, $\tilde{F}$ is a τ_i-locally finite collection in X. So, there exists a τ_j-open point finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of X such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. So, $F_\alpha \cap A \subseteq G_\alpha \cap A$, for each $\alpha \in \Delta$. But $F_\alpha \cap A = F_\alpha$ since $F_\alpha \subseteq A$. So, the collection $\tilde{G}_1 = \{G_\alpha \cap A : \alpha \in \Delta\}$ is a τ_j-point finite collection of open subsets of A that expands $\tilde{F}$. Therefore, A is P-almost expandable.*

Theorem 2.3 *If X is P-almost expandable and if every P-open subset is P-almost expandable, then every subset of X is so.*

Proof 2.4 *Let S be a subset of a P-almost expandable space X and let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of subsets of S. Let $x \in S$ and $U(x)$ be an τ_i-open set containing x and intersecting only finitely many members of $\tilde{F}$. Define $U = \bigcup_{x \in S} U(x)$. So, $S \subseteq U$ and U is a τ_i-open subset of X. Since $\tilde{F}$ is a τ_i-locally finite collection in U, where U by assumption is P-almost expandable, then there exists a τ_j-open point finite collection, say $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ such that $F_\alpha \subseteq G_\alpha$. Then, the collection $\{G_\alpha \cap S : \alpha \in \Delta\}$ is a τ_j-point finite open collection of subsets of S and $F_\alpha \subseteq G_\alpha \cap S$, for each $\alpha \in \Delta$.*

Theorem 2.5 *Let $f_1 : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P-closed and P-continuous map from a bitopological space X onto a bitopological space Y. If X is P-almost expandable, then Y is so.*

Proof 2.6 *Let $i = 1, 2$ and let $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a τ_i-locally finite collection of subsets of Y. Then, by Lemma 1.12, we have $f^{-1}(\tilde{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a σ_i-locally finite collection of subsets of X. Hence, there exists a τ_i-point finite open collection in X, say $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ such that $f^{-1}(F_\alpha) \subseteq G_\alpha$, for each $\alpha \in \Delta$. If $V_\alpha = Y - f(X - G_\alpha)$, then V_α is a P-open subset of Y and $F_\alpha \subseteq V_\alpha$, for each $\alpha \in \Delta$. Also, we have $\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ is σ_i-point finite, since $y \in V_\alpha$ if and only if $f^{-1}(y) \subseteq G_\alpha$. Therefore, Y is P-almost expandable.*

Corollary 2.7 *Let $f_1 : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P-countably perfect map from a bitopological space X onto a bitopological space Y. Then, X is P-almost expandable if and only if Y is so.*

Proof 2.8 *The proof follows immediately from the previous discussion.*

3 Open Problem

From the perspective of the presented theories, one might further investigate and explore certain conditions that can guarantee the almost expandable spaces to become expandable spaces.

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