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An inequality involving medians and sides of an acute triangle

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Abstract

With the help of Maple, we establish a new inequality involving the medians and side lengths of an acute triangle. We also propose two generalized conjectures for the main result. In addition, we present a conjectured inequality, from which the main result of this paper could be derived conveniently.

Keywords: *Acute (non-obtuse) triangles; Medians; Side lengths; Walker's inequality.*

1 Introduction and Main Result

Let ABC be a triangle with the side lengths a, b, c and the corresponding medians m_a, m_b, m_c. There are many inequalities among a, b, c and m_a, m_b, m_c in the literature. For example, from the excellent monograph [3], we know that for any triangle ABC the following ratio-type inequalities hold:

$$\sum \frac{m_a^2}{bc} \geq \frac{9}{4}, \tag{1}$$

$$\sum \frac{a^2}{m_b m_c} \geq 4, \tag{2}$$

$$\sum \frac{m_a^2}{b^2+c^2} \leq \frac{9}{8}, \tag{3}$$

$$\sum \frac{a^2}{m_b^2+m_c^2} \leq 2, \tag{4}$$

where $\sum$ denote cyclic sums.

Motivated by the last inequality, the author finds the main result of this paper, i.e., the following acute triangle inequality:

Theorem 1.1 *Let ABC be an acute (non-obtuse) triangle. Then*

$$\sum \frac{a}{m_b + m_c} \leq \sqrt{3}, \tag{5}$$

with equality if and only if the acute triangle ABC is equilateral.

Inequality (5) is very compact. However, it is not easy to prove it. The author has already obtained two proofs, but both of them are complicated. Here, we only introduce one of them.

2 Lemmas

We first prove a self-symmetry inequality involving the lower bound of m_b+m_c for the acute triangle ABC.

Lemma 2.1 *Let ABC be an acute triangle ABC. Then*

$$(m_b + m_c)^2 \geq \frac{128a^4 + (33b^2 + 62bc + 33c^2)a^2 - (17b^2 - 42bc + 17c^2)(b + c)^2}{32(2a^2 + bc)}, \tag{6}$$

with equality if and only if $b = c$.

Proof Using the known identity

$$4(m_b^2 + m_c^2) = 4a^2 + b^2 + c^2, \tag{7}$$

one sees that inequality (6) is equivalent to

$$8m_b m_c \geq \frac{128a^4 + (33b^2 + 62bc + 33c^2)a^2 - (17b^2 - 42bc + 17c^2)(b + c)^2}{8(2a^2 + bc)} - (4a^2 + b^2 + c^2).$$

Multiplying both sides by 8 and simplifying gives

$$64m_b m_c \geq \frac{64a^4 + (17b^2 + 30bc + 17c^2)a^2 + 50bb^2c^2 - 17bb^4 - 17c^4}{2a^2 + bc}, \tag{8}$$

which is needed to prove.

We now first show that

$$64a^4 + (17b^2 + 30bc + 17c^2)a^2 - 17b^4 + 50b^2c^2 - 17c^4 > 0. \tag{9}$$

Putting $b+c-a=2x, c+a-b=2y, a+b-c=2z$, then we have $a=y+z, b=z+x, c=x+y$. Substituting them into (9) gives equivalent algebraic inequality:

$$\begin{aligned}&16x^4+32(y+z)x^3+(12y^2+328yz+12z^2)x^2\\&-4(y+z)(y^2-74yz+z^2)x+16(y+2z)^2(2y+z)^2>0,\end{aligned}\tag{10}$$

where $x, y, z>0$. It is clear that we only need to show

$$\begin{aligned}&(12y^2+328yz+12z^2)x^2-4(y+z)(y^2-74yz+z^2)x\\&+16(y+2z)^2(2y+z)^2>0.\end{aligned}\tag{11}$$

The left hand is a quadratic function of x, and it is easy to know that its discriminant F_x is given by

$$F_x=-3056(y^6+z^6)-101664yz(y^4+z^4)-365328y^2z^2(y^2+z^2).$$

Hence $F_x<0$ and inequality (11) holds true. And inequality (9) is proved.

Next, we shall prove inequality (8). Since we have (9), for proving (8) we only need to show that

$$\begin{aligned}Q_0\equiv&\left[16m_bm_c(2a^2+bc)^2\right]^2\\&-\left[64a^4+(17b^2+30bc+17c^2)a^2-17b^4+50b^2c^2-17c^4\right]^2\geq 0.\end{aligned}\tag{12}$$

By using the formulas $4m_b^2=2(c^2+a^2)-b^2$ and $4m_c^2=2(a^2+b^2)-c^2$, it is easy to verify the following identity:

$$\begin{aligned}Q_0=&(b-c)^2(a+b+c)(b+c-a)\left[128a^4+(289b^2-450bc+289c^2)a^2\right.\\&\left.-289b^4+610b^2c^2-289c^4\right].\end{aligned}\tag{13}$$

So, it remains to show that the strict inequality

$$128a^4+(289b^2-450bc+289c^2)a^2-289b^4+610b^2c^2-289c^4>0$$

holds for the acute $\triangle ABC$. Again, note that $225b^2-450bc+225c^2>0$, we only to prove

$$128a^4+264(b^2+c^2)a^2-289b^4+610b^2c^2-289c^4>0.\tag{14}$$

Putting $b^2+c^2-a^2=2x, c^2+a^2-b^2=2y, a^2+b^2-c^2=2z$, then $a^2=y+z, b^2=z+x, c^2=x+y (x, y, z>0)$. Substituting the later three relations into (14) and simplifying gives the equivalent inequality

$$32x^2+560(y+z)x+103y^2+1394yz+103z^2>0,$$

which is obviously true. Thus, inequalities (14), (8) and (6) are proved. Form identity (13), we conclude that the equality in (6) occurs if and only if $b=c$. This completes the proof of Lemma 2.1. □

In what follows, we denote the circumradius, inradius and semi-perimeter of the triangle ABC by R, r, s, respectively. In addition, we denote cyclic products for three triples by $\prod$.

Lemma 2.2 *In any triangle ABC, we have*

$$\prod(2s+a)=2s(9s^2+6Rr+r^2). \tag{15}$$

Proof. Note that

$$\prod(2s+a)=8s^3+4s^2\sum a+2s\sum bc+abc.$$

Then, using the following three basic identities in the triangle ABC:

$$\sum a=2s, \tag{16}$$

$$abc=4Rrs, \tag{17}$$

$$\sum bc=s^2+4Rr+r^2, \tag{18}$$

one obtains (15) immediately. □

We now point out that by using (16)-(18) one can easily obtain the following identities (see e.g.[3, pp.52-55]):

$$\sum a^2=2s^2-8Rr-2r^2, \tag{19}$$

$$\sum a^3=2s^3-(12Rr+6r^2)s, \tag{20}$$

$$\sum a^4=2s^4-4(4R+3r)rs^2+2(4R+r)^2r^2, \tag{21}$$

$$\sum b^2c^2=s^4-2(4R-r)rs^2+(4R+r)^2r^2. \tag{22}$$

Further, by applying identities (16)-(22) we can prove the following identities given in the next Lemma 2.3 and 2.4 (cf. [5] and [6]).

Lemma 2.3 *In any triangle ABC, the following identities hold:*

$$\sum b^3c^3=s^6+(-12Rr+3r^2)s^4+3r^4s^2+(4R+r)^3r^3, \tag{23}$$

$$\begin{aligned}\sum b^4c^4=&s^8+(-16Rr+4r^2)s^6+(32R^2r^2-16Rr^3+6r^4)s^4\\&+(16Rr^5+4r^6)s^2+r^4(4R+r)^4,\end{aligned} \tag{24}$$

$$\begin{aligned}\sum b^5c^5=&s^{10}+(-20Rr+5r^2)s^8+(80R^2r^2-40Rr^3+10r^4)s^6+10r^6s^4\\&+(80R^2r^6+40Rr^7+5r^8)s^2+r^5(4R+r)^5,\end{aligned} \tag{25}$$

$$\begin{aligned}\sum b^6c^6=&s^{12}-6(4R-r)rs^{10}+3(48R^2-24Rr+5r^2)r^2s^8\\&-4(32R^3-24R^2r+12Rr^2-5r^3)r^3s^6+3(16R+5r)r^7s^4\\&+6(4R+r)^3r^7s^2+(4R+r)^6r^6,\end{aligned} \tag{26}$$

$$\begin{aligned}\sum b^7c^7=&s^{14}-7(4R-r)rs^{12}+7(32R^2-16Rr+3r^2)r^2s^{10}\\&-7(64R^3-48R^2r+20Rr^2-5r^3)r^3s^8+35r^8s^6\\&+7(8R+3r)(4R+r)r^8s^4+7(4R+r)^4r^8s^2+(4R+r)^7r^7.\end{aligned} \tag{27}$$

Lemma 2.4 *In any triangle ABC, the following identities hold:*

$$\sum a^5 = 2s^5 - 20(Rr + r^2)s^3 + 10(2R + r)(4R + r)r^2 s, \tag{28}$$

$$\sum a^6 = 2s^6 - (24Rr + 30r^2)s^4 + (144R^2r^2 + 144Rr^3 + 30r^4)s^2 - 2(4R + r)^3r^3, \tag{29}$$

$$\sum a^7 = 2s^7 - 14(2R + 3r)rs^5 + 14(16R^2 + 20Rr + 5r^2)r^2s^3 - 14(2R + r)(4R + r)^2r^3s, \tag{30}$$

$$\sum a^8 = 2s^8 - (32Rr + 56r^2)s^6 + (320R^2r^2 + 480Rr^3 + 140r^4)s^4 - (1024R^3r^3 + 1280R^2r^4 + 480Rr^5 + 56r^6)s^2 + 2(4R + r)^4r^4, \tag{31}$$

$$\sum a^9 = 2s^9 - 36(R + 2r)rs^7 + 36(12R^2 + 21Rr + 7r^2)r^2s^5 - 12(160R^3 + 240R^2r + 105Rr^2 + 14r^3)r^3s^3 + 18(2R + r)(4R + r)^3r^4s, \tag{32}$$

$$\sum a^{10} = 2s^{10} - 10(4R + 9r)rs^8 + 140(2R + 3r)(2R + r)r^2s^6 - 20(160R^3 + 280R^2r + 140Rr^2 + 21r^3)r^3s^4 + 10(40R^2 + 40Rr + 9r^2)(4R + r)^2r^4s^2 - 2(4R + r)^5r^5, \tag{33}$$

$$\sum a^{11} = 2s^{11} - 22(2R + 5r)rs^9 + 44(16R^2 + 36Rr + 15r^2)r^2s^7 - 308(2R + r)(8R^2 + 12Rr + 3r^2)r^3s^5 + 22(4R + r)(160R^3 + 240R^2r + 108Rr^2 + 15r^3)r^4s^3 - (22(2R + r))(4R + r)^4r^5s, \tag{34}$$

$$\sum a^{12} = 2s^{12} - (48Rr + 132r^2)s^{10} + (864R^2r^2 + 2160Rr^3 + 990r^4)s^8 - (7168R^3r^3 + 16128R^2r^4 + 10080Rr^5 + 1848r^6)s^6 + (26880R^4r^4 + 53760R^3r^5 + 36288R^2r^6 + 10080Rr^7 + 990r^8)s^4 - 12(48R^2 + 48Rr + 11r^2)(4R + r)^3r^5s^2 + 2r^6(4R + r)^6, \tag{35}$$

$$\sum a^{13} = 2s^{13} - 52(R + 3r)rs^{11} + 130(8R^2 + 22Rr + 11r^2)r^2s^9 - 312(32R^3 + 80R^2r + 55Rr^2 + 11r^3)r^3s^7 + 26(1792R^4 + 4032R^3r + 3024R^2r^2 + 924Rr^3 + 99r^4)r^4s^5 - 52(112R^3 + 168R^2r + 77Rr^2 + 11r^3)(4R + r)^2r^5s^3 + 26(2R + r)(4R + r)^5r^6s, \tag{36}$$

$$\sum a^{14} = 2s^{14} - 14(4R + 13r)rs^{12} + 154(8R^2 + 24Rr + 13r^2)r^2s^{10} - 42(320R^3 + 880R^2r + 660Rr^2 + 143r^3)r^3s^8 + 42(1792R^4 + 4480R^3r + 3696R^2r^2 + 1232Rr^3 + 143r^4)r^4s^6 - 14(4R + r)(3584R^4 + 7168R^3r + 4928R^2r^2 + 1408Rr^3 + 143r^4)r^5s^4 + 14(56R^2 + 56Rr + 13r^2)(4R + r)^4r^6s^2 - 2(4R + r)^7r^7. \tag{37}$$

Lemma 2.5 *In any triangle ABC, let*

$$\begin{aligned}
N_1 =&a^2(2a^2+bc),\ \ N_2=b^2(2b^2+ca),\ \ N_3=c^2(2c^2+ab),\\
M_1 =&(2s+a)\left[128a^4+(33b^2+62bc+33c^2)a^2\right.\\
&\left.-(17b^2-42bc+17c^2)(b+c)^2\right],\\
M_2 =&(2s+b)\left[128b^4+(33c^2+62ca+33a^2)b^2\right.\\
&\left.-(17c^2-42ca+17a^2)(c+a)^2\right],\\
M_3 =&(2s+c)\left[128c^4+(33a^2+62ab+33b^2)c^2\right.\\
&\left.-(17a^2-42ab+17b^2)(a+b)^2\right].
\end{aligned}$$

Then

$$M_1M_2M_3=512s^3(9s^2+6Rr+r^2)K_1, \tag{38}$$

$$N_1M_2M_3+N_2M_3M_1+N_3M_1M_2=256s^2K_2, \tag{39}$$

where

$$\begin{aligned}
K_1 =&2592s^{10}-2(26064R-22537r)rs^8+3(99072R^2-195928Rr\\
&+53453r^2)r^2s^6-36(11088R^3-44705R^2r+22969Rr^2\\
&+8382r^3)r^3s^4+(540288R^4-2415104R^3r+231384R^2r^2\\
&+847236Rr^3+196867r^4)r^4s^2+1296(72R^2+76Rr\\
&+19r^2)(4R+r)^3r^5,\\
K_2 =&432s^{12}-4(2022R-1667r)rs^{10}+2(25192R^2-37718Rr\\
&+11593r^2)r^2s^8-(134688R^3-182993R^2r+102660Rr^2\\
&+9650r^3)r^3s^6+2(133680R^4-261078R^3r-87527R^2r^2\\
&+18110Rr^3+7267r^4)r^4s^4+(4R+r)(446416R^4+553736R^3r\\
&+274017R^2r^2+55252Rr^3+4142r^4)r^5s^2+1296(4R+r)^4R^2r^6.
\end{aligned}$$

Proof. Firstly, we prove identity (38). Let

$$E_0=\prod\left[128a^4+(33b^2+62bc+33c^2)a^2-(17b^2-42bc+17c^2)(b+c)^2\right]. \tag{40}$$

With the help of Maple, we easily obtain

$$\begin{aligned}
E_0 =&36992a^{12}-17408a^{11}b-17408a^{11}c-171071a^{10}b^2+26110a^{10}bc\\
&-171071a^{10}c^2+11896a^9b^3-96632a^9b^2c-96632a^9bc^2\\
&+11896a^9c^3-81812a^8b^4-11378a^8b^3c+241068a^8b^2c^2\\
&-11378a^8bc^3-81812a^8c^4+171400a^7b^5+366970a^7b^4c\\
&+281086a^7b^3c^2+281086a^7b^2c^3+366970a^7bc^4+171400a^7c^5
\end{aligned} \tag{41}$$

$$
\begin{aligned}
&+ 763558a^6b^6 + 54898a^6b^5c + 844723a^6b^4c^2 + 670762a^6b^3c^3 \\
&+ 844723a^6b^2c^4 + 54898a^6bc^5 + 763558a^6c^6 + 171400a^5b^7 \\
&+ 54898a^5b^6c + 1377588a^5b^5c^2 + 560658a^5b^4c^3 + 560658a^5b^3c^4 \\
&+ 1377588a^5b^2c^5 + 54898a^5bc^6 + 171400a^5c^7 - 81812a^4b^8 \\
&+ 366970a^4b^7c + 844723a^4b^6c^2 + 560658a^4b^5c^3 + 3059658a^4b^4c^4 \\
&+ 560658a^4b^3c^5 + 844723a^4b^2c^6 + 366970a^4bc^7 - 81812a^4c^8 \\
&+ 11896a^3b^9 - 11378a^3b^8c + 281086a^3b^7c^2 + 670762a^3b^6c^3 \\
&+ 560658a^3b^5c^4 + 560658a^3b^4c^5 + 670762a^3b^3c^6 + 281086a^3b^2c^7 \\
&- 11378a^3bc^8 + 11896a^3c^9 - 171071a^2b^{10} - 96632a^2b^9c \\
&+ 241068a^2b^8c^2 + 281086a^2b^7c^3 + 844723a^2b^6c^4 + 1377588a^2b^5c^5 \\
&+ 844723a^2b^4c^6 + 281086a^2b^3c^7 + 241068a^2b^2c^8 - 96632a^2bc^9 \\
&- 171071a^2c^{10} - 17408ab^{11} + 26110ab^{10}c - 96632ab^9c^2 \\
&- 11378ab^8c^3 + 366970ab^7c^4 + 54898ab^6c^5 + 54898ab^5c^6 \\
&+ 366970ab^4c^7 - 11378ab^3c^8 - 96632ab^2c^9 + 26110abc^{10} \\
&- 17408ac^{11} + 36992b^{12} - 17408b^{11}c - 171071b^{10}c^2 \\
&+ 11896b^9c^3 - 81812b^8c^4 + 171400b^7c^5 + 763558b^6c^6 \\
&+ 171400b^5c^7 - 81812b^4c^8 + 11896b^3c^9 - 171071b^2c^{10} \\
&- 17408bc^{11} + 36992c^{12}.
\end{aligned}
$$

Then, we can further obtain

$$
\begin{aligned}
E_0 = &- 111756\sum a^{12} - 17408\sum a\sum a^{11} - 287748abc\sum a^9 \\
&- 96632abc\sum a\sum a^8 + 11896\sum a^3\sum a^9 \\
&- 81812\sum b^4c^4\sum a^4 - 11378abc\sum a^2\sum a^7 \\
&+ 17140\sum a^2\sum b^5c^5 + 1360448(abc)^2\sum b^3c^3 \\
&+ 366970abc\sum a^3\sum a^6 + 154260\sum a^5\sum a^7 \\
&+ 54898abc\sum a^4\sum a^5 - 171071\sum b^2c^2\sum a^8 \\
&- 713670(abc)^2\sum a^6 + 110104(abc)^3\sum a^3 \\
&+ 560658(abc)^3\sum a\sum a^2 + 844723(abc)^2\sum a^2\sum a^4 \\
&+ 281086(abc)^2\sum a\sum a^5 + 763558\sum b^6c^6 + 3305094a^4b^4c^4. \qquad (42)
\end{aligned}
$$

Again, by using the previous identities (16)-(22) and some identities given in Lemma 2.3 and 2.4, we obtain the following identity:

$$E_0 = 256s^2K_1. \qquad (43)$$

Since

$$M_1M_2M_3 = E_0\prod(2s+a),$$

then identity (38) follows from (43) and (15) immediately.

Secondly, we prove identity (39). Let

$$F_0 = N_1M_2M_3 + N_2M_3M_1 + N_3M_1M_2.$$

With the help of software Maple, we easily get

$$\begin{aligned}
F_0 =&578a^{14} + 1462a^{13}b + 1462a^{13}c - 2482a^{12}b^2 + 1675a^{12}bc\\
&- 2482a^{12}c^2 - 8754a^{11}b^3 - 14915a^{11}b^2c - 14915a^{11}bc^2\\
&- 8754a^{11}c^3 - 14054a^{10}b^4 - 21611a^{10}b^3c - 34629a^{10}b^2c^2\\
&- 21611a^{10}bc^3 - 14054a^{10}c^4 - 7294a^9b^5 - 7033a^9b^4c\\
&+ 8749a^9b^3c^2 + 8749a^9b^2c^3 - 7033a^9bc^4 - 7294a^9c^5\\
&+ 43606a^8b^6 + 37784a^8b^5c + 107205a^8b^4c^2 + 111039a^8b^3c^3\\
&+ 107205a^8b^2c^4 + 37784a^8bc^5 + 43606a^8c^6 + 84468a^7b^7\\
&+ 130894a^7b^6c + 226454a^7b^5c^2 + 263027a^7b^4c^3 + 263027a^7b^3c^4\\
&+ 226454a^7b^2c^5 + 130894a^7bc^6 + 84468a^7c^7 + 43606a^6b^8\\
&+ 130894a^6b^7c + 321124a^6b^6c^2 + 411102a^6b^5c^3 + 491977a^6b^4c^4\\
&+ 411102a^6b^3c^5 + 321124a^6b^2c^6 + 130894a^6bc^7 + 43606a^6c^8\\
&- 7294a^5b^9 + 37784a^5b^8c + 226454a^5b^7c^2 + 411102a^5b^6c^3\\
&+ 696800a^5b^5c^4 + 696800a^5b^4c^5 + 411102a^5b^3c^6 + 226454a^5b^2c^7\\
&+ 37784a^5bc^8 - 7294a^5c^9 - 14054a^4b^{10} - 7033a^4b^9c\\
&+ 107205a^4b^8c^2 + 263027a^4b^7c^3 + 491977a^4b^6c^4 + 696800a^4b^5c^5\\
&+ 491977a^4b^4c^6 + 263027a^4b^3c^7 + 107205a^4b^2c^8 - 7033a^4bc^9\\
&- 14054a^4c^{10} - 8754a^3b^{11} - 21611a^3b^{10}c + 8749a^3b^9c^2\\
&+ 111039a^3b^8c^3 + 263027a^3b^7c^4 + 411102a^3b^6c^5 + 411102a^3b^5c^6\\
&+ 263027a^3b^4c^7 + 111039a^3b^3c^8 + 8749a^3b^2c^9 - 21611a^3bc^{10}\\
&- 8754a^3c^{11} - 2482a^2b^{12} - 14915a^2b^{11}c - 34629a^2b^{10}c^2\\
&+ 8749a^2b^9c^3 + 107205a^2b^8c^4 + 226454a^2b^7c^5 + 321124a^2b^6c^6\\
&+ 226454a^2b^5c^7 + 107205a^2b^4c^8 + 8749a^2b^3c^9 - 34629a^2b^2c^{10}\\
&- 14915a^2bc^{11} - 2482a^2c^{12} + 1462ab^{13} + 1675ab^{12}c - 14915ab^{11}c^2\\
&- 21611ab^{10}c^3 - 7033ab^9c^4 + 37784ab^8c^5 + 130894ab^7c^6\\
&+ 130894ab^6c^7 + 37784ab^5c^8 - 7033ab^4c^9 - 21611ab^3c^{10}\\
&- 14915ab^2c^{11} + 1675abc^{12} + 1462ac^{13} + 578b^{14} + 1462b^{13}c \qquad (44)
\end{aligned}$$

$$- 2482b^{12}c^2 - 8754b^{11}c^3 - 14054b^{10}c^4 - 7294b^9c^5 + 43606b^8c^6$$
$$+ 84468b^7c^7 + 43606b^6c^8 - 7294b^5c^9 - 14054b^4c^{10} - 8754b^3c^{11}$$
$$- 2482b^2c^{12} + 1462bc^{13} + 578c^{14}.$$

From the above identity, one easily obtains that

$$\begin{aligned} F_0 =& 17114\sum a^{14} + 84468\sum b^7c^7 + 69680(abc)^4\sum bc \\ &+ 491977(abc)^4\sum a^2 - 554336(abc)^3\sum a^5 + 277518(abc)^2\sum b^4c^4 \\ &+ 1462\sum bc\sum a^{12} - 124906abc\sum a^{11} - 7294\sum b^5c^5\sum a^4 \\ &+ 634414abc^4\sum bc + 43606\sum a^2\sum b^6c^6 - 2482\sum a^2\sum a^{12} \\ &- 8754\sum b^3c^3\sum a^8 - 377037(abc)^2\sum a^8 + 263027(abc)^3\sum a\sum a^4 \\ &+ 411102(abc)^3\sum a^2\sum a^3 + 75568abc\sum a^4\sum a^7 \\ &- 7033abc\sum a^3\sum a^8 + 107205(abc)^2\sum a^2\sum a^6 \\ &+ 226454a^2b^2c^2\sum a^3\sum a^5 + 130894abc\sum a^5\sum a^6 \\ &- 21611abc\sum a^2\sum a^9 - 37784abc\sum a^4\sum a^7 \\ &+ 8749(abc)^2\sum a\sum a^7 - 14915abc\sum a\sum a^{10} \\ &- 14054\sum a^4\sum a^{10}. \end{aligned} \tag{45}$$

Further, making use of Maple and using the previous identities, we obtain $F_0 = 256s^2K_2$, i.e., identity (39). This completes the proof os Lemma 2.5. □

Lemma 2.6 *In an acute (non-obtuse) triangle ABC, we have*

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{46}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Inequality (46) was first proposed by A.W.Walker in [1]. Recently, the author obtained a general generalization of Walker's inequality in [7].

Lemma 2.7 *In an acute (non-obtuse) triangle ABC, we have*

$$s \geq 2R + r, \tag{47}$$

with equality if and only if $\triangle ABC$ is right.

Inequality (47) was first given in [2] by C.Ciamberlini.

Lemma 2.8 *In an acute (non-obtuse) triangle ABC, we have*

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}, \tag{48}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Inequality (48) was established by the author in a Chinese paper [4]. A direct proof of (48) was given by the author in [7].

3 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1.

Proof Applying the Cauchy inequality, we have

$$\left(\sum \frac{a}{m_b + m_c}\right)^2 \leq \sum \frac{a^2}{(2s+a)(m_b+m_c)^2} \sum (2s+a).$$

Note that $\sum(2s+a) = 8s$. Thus, to prove inequality (5) we only need to prove

$$\sum \frac{a^2}{(2s+a)(m_b+m_c)^2} \leq \frac{3}{8s}. \tag{49}$$

By Lemma 2.1 and Lemma 2.5, we have

$$\sum \frac{a^2}{(2s+a)(m_b+m_c)^2} \leq 32\left(\frac{N_1}{M_1} + \frac{N_2}{M_2} + \frac{N_3}{M_3}\right). \tag{50}$$

So, we need to prove that

$$\frac{N_1}{M_1} + \frac{N_2}{M_2} + \frac{N_3}{M_3} \leq \frac{3}{256s}. \tag{51}$$

In view of identities (38) and (39), one knows that the above inequality is equivalent to

$$\frac{K_2}{(9s^2+6Rr+r^2)K_1} \leq \frac{3}{128},$$

that is

$$3K_1(9s^2+6Rr+r^2) - 128K_2 \geq 0. \tag{52}$$

Putting

$$3K_1(9s^2+6Rr+r^2) - 128K_2 = P_0$$

and using the expressions of K_1 and K_2 given in Lemma 2.5, we obtain

$$\begin{aligned}P_0 =&14688s^{12} - 2(162768R - 185635r)rs^{10} + (637376R^2 - 5559412Rr\\&+ 1497107r^2)r^2s^8 + (11812416R^3 + 10341692R^2r - 8062278Rr^2\\&- 6431027r^3)r^3s^6 - (26819328R^4 - 29399496R^3r - 18598508R^2r^2\\&- 10327024Rr^3 - 2549801r^4)r^4s^4 - (57596672R^5 + 91371648R^4r\\&+ 13821136R^3r^2 - 18926032R^2r^3 - 7529922Rr^4 - 725273r^5)r^5s^2\\&+ 1296(784R^3 + 1456R^2r + 570Rr^2 + 57r^3)(4R + r)^3r^6. \qquad (53)\end{aligned}$$

We have to prove inequality $P_0 \geq 0$. Through analysis, we rewritten P_0 as follows:

$$\begin{aligned}P_0 \equiv&14688w_0^6 + (176256R^2 + 379488Rr + 635654r^2)w_0^5\\&+ (881280R^4 + 3794880R^3r + 8072956R^2r^2 + 14983708Rr^3\\&+ 9049037r^4)w_0^4 + (2350080R^6 + 15179520R^5r + 39157488R^4r^2\\&+ 94140192R^3r^3 + 163042596R^2r^4 + 185495722Rr^5\\&+ 52880077r^6)w_0^3 + (3525120R^8 + 30359040R^7r + 92046304R^6r^2\\&+ 205232160R^5r^3 + 498196128R^4r^4 + 925097380R^3r^5\\&+ 1440439622R^2r^6 + 818785402Rr^7 + 143603156r^8)w_0^2\\&+ (2820096R^{10} + 30359040R^9r + 105777632R^8r^2\\&+ 170724992R^7r^3 + 325842544R^6r^4 + 781865880R^5r^5\\&+ 2132461180R^4r^6 + 4644448392R^3r^7 + 4000583644R^2r^8\\&+ 1417202480Rr^9 + 175853360r^{10})w_0 + 8(R - 2r)(117504R^{11}\\&+ 1752960R^{10}r + 9481368R^9r^2 + 23200680R^8r^3\\&+ 23708454R^7r^4 - 34169590R^6r^5 - 236354339R^5r^6\\&- 479517121R^4r^7 - 453717541R^3r^8 - 217182551R^2r^9\\&- 50860120Rr^{10} - 4609500r^{11}), \qquad (54)\end{aligned}$$

where $w_0 = s^2 - 2R^2 - 8Rr - 3r^2$. Thus, according to Walker's acute triangle inequality (46), i.e. $w_0 \geq 0$, it remains to prove that

$$\begin{aligned}X_0 \equiv&(2820096R^{10} + 30359040R^9r + 105777632R^8r^2\\&+ 170724992R^7r^3 + 325842544R^6r^4 + 781865880R^5r^5\\&+ 2132461180R^4r^6 + 4644448392R^3r^7 + 4000583644R^2r^8\\&+ 1417202480Rr^9 + 175853360r^{10})w_0 + 8(R - 2r)(117504R^{11}\\&+ 1752960R^{10}r + 9481368R^9r^2 + 23200680R^8r^3\\&+ 23708454R^7r^4 - 34169590R^6r^5 - 236354339R^5r^6\\&- 479517121R^4r^7 - 453717541R^3r^8 - 217182551R^2r^9\\&- 50860120Rr^{10} - 4609500r^{11}) \geq 0. \qquad (55)\end{aligned}$$

Substituting $w_0 = s^2 - 2R^2 - 8Rr - 3r^2$ into (55), we know that $X_0 \geq 0$ is equivalent to

$$X_0 \equiv G_0 s^2 - H_0 \geq 0, \tag{56}$$

where

$$\begin{aligned} G_0 =& 2820096R^{10} + 30359040R^9 r + 105777632R^8 r^2 + 170724992R^7 r^3 \\ &+ 325842544R^6 r^4 + 781865880R^5 r^5 + 2132461180R^4 r^6 \\ &+ 4644448392R^3 r^7 + 4000583644R^2 r^8 + 1417202480Rr^9 \\ &+ 175853360r^{10} \\ H_0 =& 4700160R^{12} + 71135232R^{11} r + 415084288R^{10} r^2 + 1244844608R^9 r^3 \\ &+ 2516361168R^8 r^4 + 5335339072R^7 r^5 + 12841498304R^6 r^6 \\ &+ 28748651408R^5 r^7 + 47511604356R^4 r^8 + 43250399040R^3 r^9 \\ &+ 20623037636R^2 r^{10} + 4881548400Rr^{11} + 453808080r^{12}. \end{aligned}$$

Next, we consider the following two cases to complete the proof of inequality (56).

Case 1 R and r satisfy $R > \dfrac{12}{5}r$.

In this case, according to Lemma 2.7, we only need to show

$$X_1 \equiv G_0(2R + r)^2 - H_0 > 0. \tag{57}$$

But it is easy to check the following identity:

$$\begin{aligned} X_1 =& 6580224R^{12} + 61581312R^{11} r + 132282496R^{10} r^2 - 108475072R^9 r^3 \\ &- 424313392R^8 r^4 - 733780384R^7 r^5 - 858347520R^6 r^6 - 859147240R^5 r^7 \\ &- 10799015032R^4 r^8 - 16934806152R^3 r^9 - 10250230632R^2 r^{10} \\ &- 2760932480Rr^{11} - 277954720r^{12}. \end{aligned} \tag{58}$$

Letting $R = \dfrac{12}{5}r + t(t \geq 0)$ and substituting it into $5^{12}X_1$, we obtain

$$\begin{aligned} & 5^{12}X_1 \\ =& 1606500000000000t^{12} + 6130170000000000t^{11} r \\ &+ 10399333712500000000t^{10} r^2 + 10397355498125000000t^9 r^3 \\ &+ 6837187429878125000 0t^8 r^4 + 3118338799209575000 00t^7 r^5 \\ &+ 101108670939936600000 0t^6 r^6 + 2343182740076958575000t^5 r^7 \\ &+ 3830709401420052425000t^4 r^8 + 4247253236660803995000t^3 r^9 \\ &+ 2908674790564705505400t^2 r^{10} + 948160220765454425920tr^{11} \\ &+ 85783879234465922 24r^{12}. \end{aligned} \tag{59}$$

So, we have $X_1 > 0$ and inequality $X_0 > 0$ is proved under the first case.

Case 2 R and r satisfy $R \leq \frac{12}{5}r$.

In this case, according to Lemma 2.8, to prove inequality (56) we need to prove

$$G_0\left(16Rr - 3r^2 - \frac{4r^3}{R}\right) - H_0 \geq 0,$$

that is

$$X_2 \equiv G_0(16R^2r - 3Rr^2 - 4r^3) - RH_0 \geq 0. \tag{60}$$

It is easy to get

$$X_2 = 8(R - 2r)J_0, \tag{61}$$

where

$$\begin{aligned} J_0 = &- 587520R^{12} - 4426752R^{11}r - 1078496R^{10}r^2 + 40998008R^9r^3 \\ &+ 54054722R^8r^4 - 24033540R^7r^5 - 297076058R^6r^6 \\ &- 378932159R^5r^7 + 1401476039R^4r^8 + 2589920749R^3r^9 \\ &+ 1613923691R^2r^{10} + 437617800Rr^{11} + 43963340r^{12}. \end{aligned}$$

Recall that for any triangle ABC we have Euler's inequality:

$$R \geq 2r. \tag{62}$$

To prove $X_2 \geq 0$ we only need to show the strict inequality $J_0 >$. By the hypothesis $r \geq 5R/12$, we may assume that $r = \frac{5}{12}R + t(t \geq 0)$. Substituting it into $5^{12}J_0$ and expanding gives

$$\begin{aligned} &5^{12}J_0 \\ =&391981555480830935040t^{12} + 586175204014895923200 0t^{11}R \\ &+ 36764813938907620048896t^{10}R^2 + 126545260028768387334144t^9R^3 \\ &+ 263931556557371086209024t^8R^4 + 350219485178890119806976t^7R^5 \\ &+ 304172049489435629125632t^6R^6 + 174813364582621808394240t^5R^7 \\ &+ 66284954461532042744832t^4R^8 + 16721181783197508768768t^3R^9 \\ &+ 3022721439862356998784t^2R^{10} + 395031041654366409408tR^{11} \\ &+ 16327486235712839900R^{12}. \end{aligned} \tag{63}$$

So, we have $J_0 > 0$ and inequality $X_2 \geq 0$ follows from (61) and (62). Thus, inequality $X_0 \geq 0$ is proved under the second case.

Combining the arguments of the above two cases, we conclude that inequality (56) holds for all acute triangles. Hence we complete the proof of inequality (5). Also, it is easily known that the equality in (5) holds if and only if the acute $\triangle ABC$ is equilateral. This completes the proof of Theorem 1.1. □

4 Three Open Problems

In this section, we propose three conjectures related to inequality (5) as open problems.

Considering generalizations of inequality (5) and verifying by the computer, we propose the following conjecture:

Conjecture 4.1 *Let k be a real number such that $-0.85 \leq k < 1.5$ or $7.48 < k \leq 11.18$. Then for the acute ABC the following inequality holds:*

$$\sum \frac{a^k}{m_b^k + m_c^k} \leq \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)^k. \tag{64}$$

If $1.5 \leq k \leq 7.48$, then the above inequality holds for any triangle ABC.

Clearly, inequality (5) is a special case of (64) when $k = 1$. Also, inequality (4) is a special case of (64) when $k = 2$.

On the other hand, we present the following generalization of inequality (5):

Conjecture 4.2 *Let k be a real number such that $-0.21 \leq k \leq 0.55$. Then for the acute ABC the following inequality holds:*

$$\sum \frac{a}{m_b + m_c + km_a} \leq \frac{2\sqrt{3}}{k+2}. \tag{65}$$

When $k = 0$, the above inequality becomes (5).

Finally, we present the following inequality related to the main result of this paper:

Conjecture 4.3 *Let ABC be an acute triangle. Then*

$$\sum \frac{a^2}{(m_b + m_c)^3} \leq \frac{3}{2\sum m_a}. \tag{66}$$

If inequality (66) holds, then inequality (5) could be derived immediately by using the Cauchy inequality.

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