*Int. J. Open Problems Compt. Math., Vol. 14, No. 4, December 2021*
*Print ISSN: 1998-6262, Online ISSN: 2079-0376*


# A Short Survey On WC–Banach Algebras

**Bilel Krichen and Bilel Mefteh**

Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax,
Soukra Road Km 3.5, B.P. 1171, 3000, Sfax-Tunisia
e-mail:bilel.krichen@fss.usf.tn, bilel.mefteh@fss.usf.tn

Received 28 September 2021; Accepted 04 November 2021

**Abstract**

***In the last years, the theory of Banach algebras has been well developed in the weak topology. In this survey we will recall some new classes of Banach algebras respecting chronological order by making links between themselves and we will finish by asking an open question.***

**Keywords:** *Condition ($\mathcal{P}$), WC–Banach algebra, RWC–Banach algebra.*
**2010 Mathematics Subject Classification:** 46H05.

## 1 Introduction

The concept of Banach algebras plays an important role in the theory of differential and integral equations (see [1, 2, 3, 4, 7, 8]). Several authors have introduced classes of Banach algebras and established fixed point results in the weak topology. In 2010, Ben Amar, Chouayekh, and Jeribi [4] introduced a class of Banach algebra satisfying the following sequential condition:

$$(\mathcal{P}) \qquad \begin{cases} \text{For any sequences } (x_n)_{n\in\mathbb{N}} \text{ and } (y_n)_{n\in\mathbb{N}} \text{ of } X \text{ such that } x_n \rightharpoonup x \\ \text{and } y_n \rightharpoonup y, \text{ then } x_n \cdot y_n \rightharpoonup x \cdot y; \text{ where } X \text{ is a Banach algebra,} \end{cases}$$

and they proved some fixed point theorems for the sum and the product of nonlinear operators.

In 2014, Banas and Taoudi [3] extended some of the results established in [4] in the weak topology setting. In particular, the authors introduced the class, probably weaker than the property ($\mathcal{P}$), of weakly compact Banach algebras (WC–Banach algebras, in short) i.e., Banach algebras in which the product of two weakly compact subsets is weakly compact. In 2015, Jeribi and Krichen [7] raised the question of whether there exists a WC–Banach algebra in which the property $(\mathcal{P})$ fails and this question remained unanswered until 2019. Banas and Olszowy [2] gave a positive answer and proved that a Banach algebra $X$ satisfies condition $(\mathcal{P})$ if, and only if, $X$ is a WC–Banach algebra. In this direction, the authors introduced a new class of Banach algebras, strictly containing that of weakly compact, the so-called relatively weakly compact Banach algebras (RWC–Banach algebras, in short) i.e., Banach algebras in which the product of two relatively weakly compact subsets is relatively weakly compact.

## 2 Preliminaries

Throughout the survey we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers. Assume that $X$ is a Banach space with the norm $\|\cdot\|$ and the zero element $\theta$ and $\mathcal{D}(A)$ denotes the domain of an operator $A$. We use the standard notation, $\rightharpoonup$ to denote weak convergence and $\to$ to denote strong convergence in $X$, respectively.

Recall from [7] that a Banach space $X$ is said to be a Banach algebra if it is endowed with the inner operation of multiplication $x \cdot y$ of elements $x, y \in X$ which is associative, bilinear, and such that

$$\|x \cdot y\| \leq \|x\|\|y\|,$$

for $x, y \in X$.
Assume that $X$ is a Banach algebra. For two arbitrary subsets $U, V$ of $X$, we define the product $U \cdot V$ in the following way:

$$U \cdot V = \{u \cdot v \ : u \in U,\ v \in V\}.$$

Now, we will give some definitions.

**Definition 2.1** *We say that a Banach space $X$ has the Dunford-Pettis Property if for each Banach space $Y$ every weakly compact linear operator $T\colon X \to Y$ maps weakly convergent sequences into strongly convergent sequences.*

**Definition 2.2** *We say that a Banach space $X$ has Schur's property if for any sequence $(x_n) \subseteq \mathcal{D}(A)$ which $x_n \rightharpoonup x$ implies that $x_n \to x$.*

## 3 Condition $(\mathcal{P})$ and WC–Banach Algebra

To overcome the lack of stability of convergence of the product sequences under the weak topology, Ben Amar, Chouayekh, and Jeribi [4] introduced in 2010 a class of Banach algebra satisfying the condition $(\mathcal{P})$ and they proved some fixed point theorems for the sum and the product of nonlinear operators.

In what follows we indicate a wide class of Banach algebras satisfying condition $(\mathcal{P})$.

**Example 3.1** *[4]* Every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. Even, if $X$ satisfies condition $(\mathcal{P})$ then $\mathcal{C}(K, X)$ is also Banach algebra satisfying condition $(\mathcal{P})$, where $K$ is a compact Hausdorff space. Indeed, let $(x_n)$ and $(y_n)$ two sequences of $\mathcal{C}(K, X)$ such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. So, for each $t \in K$, and using the Dobrakov theorem [5] we have $x_n(t) \rightharpoonup x(t)$ and $y_n(t) \rightharpoonup y(t)$. Since $X$ verify condition $(\mathcal{P})$, then

$$x_n(t) \cdot y_n(t) \rightharpoonup x(t) \cdot y(t).$$

because $(x_n \cdot y_n)$ is a bounded sequence, this further implies that

$$x_n \cdot y_n \rightharpoonup x \cdot y,$$

which shows that the space $\mathcal{C}(K, X)$ verifies condition $(\mathcal{P})$.

**Example 3.2** *[3]* Let $X$ be a commutative Banach algebra with Dunford-Pettis Property [7]. Then, $X$ satisfies condition $(\mathcal{P})$.

**Example 3.3** Let $X$ be a Banach algebra with Schur's property. Then, $X$ satisfies condition $(\mathcal{P})$.

**Theorem 3.4** *[2]* A Banach algebra $X$ satisfies condition $(\mathcal{P})$ if and only if, for arbitrary sequences $(u_n)$ and $(v_n)$ in $X$ such that $u_n \rightharpoonup \theta$ and $v_n \rightharpoonup \theta$, we have that $u_n \cdot v_n \rightharpoonup \theta$.

Knowing that the product $W \cdot W'$ of two arbitrary weakly compact subsets $W$, $W'$ of a Banach algebra $X$ is not necessarily weakly compact. In 2014, Banas and Taoudi [3] extended some of the results established in [4] in the weak topology setting. In particular, the authors introduced the class, probably weaker than the property $(\mathcal{P})$, of weakly compact Banach algebras (WC–Banach algebras, in short) i.e., Banach algebras in which the product of two weakly compact subsets is weakly compact.
We give some examples and the link between the condition $(\mathcal{P})$ and the weakly compact Banach algebras.

**Example 3.5** *[3]* Clearly, every finite dimensional Banach algebra is a WC–Banach algebra.
Let $K$ be a Hausdorff compact space. For a given Banach algebra $X$ denote by $C(K, X)$ the Banach algebra of all continuous functions acting from $K$ into

$X$ and equipped with the supremum norm. It may be shown that $C(K,X)$ forms the WC–Banach algebra, provided $X$ is a WC–Banach algebra. Indeed, utilizing the Dobrakov theorem [5] characterizing the weak convergence in the Banach space $C(K,X)$ we can show that $C(K,X)$ is the WC–Banach algebra.

**Example 3.6** *[1]* Now, let us consider the classical Banach sequence space $c_0$ consisting of all real (or complex) sequence converging to zero and equipped with the standard supremum (maximum) norm with the product of two sequences $x=(x_n)$, $y=(y_n)\in c_0$ in the classical way:

$$x\cdot y=(x_n)\cdot(y_n)=(x_ny_n).$$

Observe that for $x,y\in c_0$ we have

$$\begin{aligned}\|x\cdot y\| &= \sup\{|x_ny_n|:\ n\in\mathbb{N}\}\\ &\le \|x\|\sup\{|y_n|:\ n\in\mathbb{N}\}\\ &= \|x\|\|y\|.\end{aligned}$$

Thus $c_0$ is a Banach algebra (normalized).
We show that $c_0$ is the WC–Banach algebra. To this end let us recall [6] that in the Banach space $c_0$ the sequence $(x_k)_{k\in\mathbb{N}}=((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ (denoted $((x_k^n))$ in the sequel), where $x_k=(x_n^k)_{n\in\mathbb{N}}\in c_0$ for any $k=1,2,\dots$, is convergent to an element $x=(x_n)_{n\in\mathbb{N}}\in c_0$ if and only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in $c_0$ and $\lim\limits_{k\to\infty}x_n^k=x_n$ for any $n=1,2,\dots$. In other words, the sequence $(x_k)_{k\in\mathbb{N}}=((x_k^n))$ is weakly convergent to $x=(x_n)_{n\in\mathbb{N}}\in c_0$ if only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in $c_0$ and is coordinatewise convergent to $x=(x_n)_{n\in\mathbb{N}}$.
Further, let us assume that $W$ and $W'$ are weakly compact subsets of the space $c_0$. Consider the product $W\cdot W'$. Let us take an arbitrary sequence $(z_k)_{k\in\mathbb{N}}\subset W\cdot W'$, $z_k=(z_n^k)_{n\in\mathbb{N}}$ for any $k=1,2,\dots$. This means that there exist two sequences $(x_k)_{k\in\mathbb{N}}\subset W$, $(y_k)_{k\in\mathbb{N}}\subset W'$ such that $z_k=x_k\cdot y_k$ for any $k=1,2,\dots$.
Since the sets $W$ and $W'$ are weakly compact, without loss of generality we can assume that $x_k\rightharpoonup x=(x_n)_{n\in\mathbb{N}}\in W$ and $y_k\rightharpoonup y=(y_n)_{n\in\mathbb{N}}\in W'$ as $k\to\infty$. If we denote $x_k=(x_n^k)_{n\in\mathbb{N}}$, $y_k=(y_n^k)_{n\in\mathbb{N}}$ for each $k=1,2,\dots$, then in view of the above quoted characterization of the weak convergence in $c_0$ we deduce that $\lim\limits_{k\to\infty}x_n^k=x_n$ for any $n=1,2,\dots$. This implies that $\lim\limits_{k\to\infty}z_n^k=\lim\limits_{k\to\infty}x_n^ky_n^k=x_ny_n$ for $n=1,2,\dots$. Obviously, the sequence $(z_k)_{k\in\mathbb{N}}=(x_k)_{k\in\mathbb{N}}\cdot(y_k)_{k\in\mathbb{N}}$ is bounded in $c_0$.
Thus we showed that the sequence $(z_k)_{k\in\mathbb{N}}$ is weakly convergent to the element $z=x\cdot y\in W\cdot W'$. This allows us to infer that the set $W\cdot W'$ is weakly compact in the space $c_0$.

**Proposition 3.1** *[3]* If $X$ is a Banach algebra satisfying condition $(\mathcal{P})$ then $X$ is a WC–Banach algebra.

In 2015, Jeribi and Krichen [7] raised the question of whether there exists a WC–Banach algebra in which the property $(\mathcal{P})$ fails and this question remained unanswered until 2019.

**Question 3.2** *[7]* Are there any examples of WC–Banach algebras which do not satisfy the condition $(\mathcal{P})$.

The previous question remained unanswered until 2019.
Recently, Banas and Olszowy [2] have shown the equivalence between WC–Banach algebra and a Banach algebra satisfying $(\mathcal{P})$.

**Theorem 3.7** *[2]* A Banach algebra $X$ satisfies condition $(\mathcal{P})$ if and only if $X$ is the WC–Banach algebra.

# 4 RWC–Banach Algebras

Banas and Olszowy [2] introduced a new class of Banach algebras, strictly containing that of weakly compact, the so-called relatively weakly compact Banach algebras (RWC–Banach algebras, in short) i.e., Banach algebras in which the product of two relatively weakly compact subsets is relatively weakly compact.
We give some examples of RWC–Banach algebras which do not satisfy the condition $(\mathcal{P})$, and we give an open question.

**Remark 4.1** *[2] If $X$ is a WC–Banach algebra, then $X$ is a RWC–Banach algebra. However, if $X$ is the RWC–Banach algebra, then $X$ is not necessarily WC–Banach algebra as can be seen in the following counter-example:*
*Consider the Banach algebra $l^2$ with the norm*

$$\|x\|_{l^2} = \|(x_n)\|_{l^2} = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$$

*and with multiplication*

$$x \cdot y = (x_n) \cdot (y_n) = \left(\sum_{n=1}^{\infty} x_n y_n, 0, 0, 0, \cdots\right).$$

*For arbitrary relatively weakly compact subsets $U$, $V$ of $l^2$, the product $U \cdot V$ is a bounded subset of the 1-dimensional subspace span$(e_1)$, where $(e_n)$ denotes the sequence of canonical vectors in $l^2$. Thus $l^2$ is a RWC–Banach algebra.*
*On the other hand, by taking $u_n = e_n$, $v_n = e_n$ for $n = 1, 2, \cdots$, we have that $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$ when $n \to \infty$. However $u_n \cdot v_n = e_1 \not\rightharpoonup 0$. Therefore, $l^2$ with the multiplication defined above is not a WC–Banach algebra.*

**Example 4.1** Every reflexive Banach algebra in particular Hilbert Banach algebra is a RWC–Banach algebra.

# 5 Open Problem

It is easy to prove that if $X$ is a WC–Banach algebra, then the Banach algebra of all continuous functions acting from $K$ into $X$ is also a WC–Banach algebra, where $K$ is a compact Hausdorff space. Hence, we can obviously deduce the following interesting open question:

**Question 5.1** *If $X$ is the RWC–Banach algebra, does the space $C(K, X)$ remain RWC–Banach algebra?*

# References

[1] J. Banas, B. Krichen and B. Mefteh, *Fixed point theorems in WC–Banach algebras and their applications to infinite systems of integral equations*, Filomat., 34:8 (2020), 2763–2784.

[2] J. Banas and L. Olszowy, *On the equivalence of some concepts in the theory of Banach algebras*, Ann. Funct. Anal., 10 (2019), 277–283.

[3] J. Banas and M. A. Taoudi, *Fixed points and solutions of operator equations for the weak topology in Banach algebras*, Taiwanese Journal of Mathematics., 18 (2014), 871–893.

[4] A. Ben Amar, S. Chouayekh and A. Jeribi, *New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations*, J. Funct. Anal., 259 (2010), 2215–2237.

[5] I. Dobrakov, *On representation of linear operators on $C_0(T, X)$*, Czechoslovak Mat. J., 21 96 (1971), 13–30.

[6] N. Dunford and J.T. Schwartz, *Linear Operators I*, International Publications, Leyden 1963.

[7] A. Jeribi and B. Krichen, *Nonlinear Functional Analysis in Banach Spaces and Banach Algebras: Fixed Point Theory under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications*, Monographs and Research Notes in Mathematics, CRC Press Taylor and Francis 2015.

[8] A. Jeribi, B. Krichen and B. Mefteh, *Fixed point theory in WC-Banach algebras*, Turk. J. Math., 40 (2016), 283–291.