

Arithmetic rough summable of I - convergence for triple sequences

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Abstract

We introduce and study some basic properties of arithmetic rough summable of I - convergent for triple sequence and also study the set of all arithmetic rough summable of I - limits for a triple sequence and relation between analytic ness and arithmetic rough summable of I - statistical convergence for a triple sequences.

Keywords: *triple sequences, rough convergence, closed and convex, cluster points and rough limit points, arithmetic summable.*

1 Introduction

A triple sequence $x = (x_{mnk})$ defined on \mathbb{N} and $u, v, w \in \mathbb{N}$, the notation $\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk}$ means the finite sum of all the numbers x_{mnk} as m, n, k ranges over the integers that divide m, n, k including 1 and m, n, k . In general for integers m, n, k and u, v, w we write $m | u, n | v$ and $k | w$ to mean m divides u , n divides v and k divides w or " m is a multiple of u , n is a multiple of v and k is a multiple of w ". We use the symbol $\langle a, u \rangle$ to denote the greatest common divisor of two integers a and u .

Now we introduced the notions arithmetic rough summability and arithmetic rough convergence as follows:

Definition 1.1 A triple sequence $x = (x_{mnk})$ defined on \mathbb{N} is called arithmetically rough summable if for each $\epsilon > 0$, and r be a positive real number there are integers u, v, w such that for every integers a, b, c we have

$$\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|(a,u)} \sum_{n|(b,v)} \sum_{k|(c,w)} x_{mnk} \right| < r + \epsilon.$$

Definition 1.2 A triple sequence $y = (y_{mnk})$ is called arithmetically rough convergent if for each $\epsilon > 0$ and r be a positive number there is an integer u, v and w such that for every integer a, b and c we have

$$|y_{abc} - y_{(a,u)(b,v)(c,w)}| < r + \epsilon.$$

The idea of rough convergence was first introduced by Phu [11, 12, 13] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r .

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the r -limit set of the sequence is equal to intersection of these sets and that r -core of the sequence is equal to the union of these sets. Dündar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of I -convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where \mathbb{N} is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough I -convergence of a triple sequence which are not studied earlier. We study the set of all rough I -limits of a triple sequence and also the relation between analytic ness and rough I -convergence of a triple sequence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ij\ell} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta(K) = \lim_{i,j,\ell \rightarrow \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where \mathbb{N}, \mathbb{R} and \mathbb{C} denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al.

[14, 15], Esi et al. [3, 4, 5, 6, 7], Dutta et al. [10], Subramanian et al. [16], Debnath et al. [8] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3 .

2 Definitions and Preliminaries

Throughout the paper \mathbb{R}^3 denotes the real three dimensional case with the metric. Consider a arithmetic rough summable of triple sequence

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ such that $x_{mnk} \in \mathbb{R}; m, n, k \in \mathbb{N}$. The following definitions are obtained:

Definition 2.1 A class I of subsets of a nonempty set $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is said to be an arithmetic summable ideal of $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, provided that

- (i) $\phi \in I$
 - (ii) $A, B \in I$ implies $A \cup B \in I$.
 - (iii) $A \in I, B \subset A$ implies $B \in I$.
- I is called a non-trivial ideal if $\phi \notin I$.

Definition 2.2 A non-empty class F of subsets of a non-empty set $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is said to be a arithmetic summable filter of $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, provided that

- (i) $\phi \in F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 2.3 Let I is a non trivial arithmetic summable ideal. Since $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi$, then the class

$F(I) =$

$M \subset \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) :$

$M = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \setminus A$ for some $A \in$

I is a filter on $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, called the arithmetic summable filter associated with I .

Definition 2.4 A non trivial ideal I in

$$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

is called arithmetic summable admissible if

$$\left\{ \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right\} \in I \text{ for each } x \in \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right).$$

Note 2.5 If I is an arithmetic summable admissible ideal, then usual convergence in $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ implies I convergence in $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$.

Remark 2.6 If I is an arithmetic summable admissible ideal, then usual rough convergence implies rough I - convergence.

Definition 2.7 Let $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ be a arithmetic summable triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be rough ideal convergent or rI - convergent to l , denoted by $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \rightarrow^{rI} l$, if for any $\epsilon > 0$ we have

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \geq r + \epsilon \right\} \in I.$$

In this case l is called rI - limit of

$$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

and a arithmetic summable triple sequence

$$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

is called rough I - convergent to l with r as roughness of degree. If $r = 0$ then it is ordinary I - convergent.

Note 2.8 Generally, a arithmetic summable triple sequence $y = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right)$ is not I - convergent in usual sense and

$$\left| \begin{array}{l} \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \\ - \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \end{array} \right| \leq r \text{ for all } (m, n, k) \in \mathbb{N}^3$$

or

$$\left\{ \left| \begin{array}{l} \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \\ - \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \end{array} \right| \geq r \right\} \in I. \text{ for some } r > 0.$$

Then the arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is rI - convergent.

Note 2.9 It is clear that rI - limit of

$$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

is not necessarily unique.

Definition 2.10 Consider rI - limit set of $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, which is denoted by $I - LIM^r_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)}$

$$\left\{ L \in X : \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \rightarrow^{rI} l \right\},$$

then the arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is said to be rI -convergent if $I - LIM^r_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \neq \phi$ and r is called a rough I - convergence degree of

$$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right).$$

Definition 2.11 A arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \in X$ is said to be I - analytic if there exists a positive real number M such that

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right|^{1/m+n+k} \geq M \right\} \in I.$$

Definition 2.12 A point $L \in X$ is said to be an I - accumulation point of a arithmetic summable triple sequence

$$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

in a metric space (X, d) if and only if for each $\epsilon > 0$ the set

$$d \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), l \right) = \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| < \epsilon \notin I. \text{ We denote the set of all } I\text{- accumulation points of}$$

$$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

$$\text{by } I \left(\Gamma \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right).$$

Definition 2.13 For a arithmetic summable triple sequence

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$I - \limsup \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = \left\{ \begin{array}{l} \sup B \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), \\ \text{if } B \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi, -\infty, \\ \text{if } B \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = \phi \end{array} \right\}, \text{ and}$$

$$I - \liminf \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = \left\{ \begin{array}{l} \inf A_x, \quad \text{if } A_x \neq \phi, \\ +\infty, \quad \text{if } A_x = \phi \end{array} \right\},$$

where

$$A_x = \left\{ a \in \mathbb{R} : \left\{ (m, n, k) \in \mathbb{N}^3 : \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) < a \right\} \notin I \right\}$$

and

$$B_x = \left\{ b \in \mathbb{R} :: \left\{ (m, n, k) \in \mathbb{N}^3 : \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) > b \right\} \notin I \right\}.$$

Definition 2.14 A arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is said to be rough I - convergent if

$I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi$. It is clear that if $I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi$ for a triple sequence $x = (x_{mnk})$ of real numbers, then we have

$$I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = [I - \limsup \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - r,$$

$$I - \liminf \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) + r].$$

3 Main Results

Theorem 3.1 Let I be an admissible ideal. For a arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, we have

$$\text{diam} \left(I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right) \leq 2r.$$

In general,

$$\text{diam} \left(I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right)$$

has an upper bound.

Proof. Assume that

$$\text{diam} \left(LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right).$$

Then, $\exists p, q \in LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \ni$:

$|p - q| > 2r$. Take $\epsilon \in \left(0, \frac{|p-q|}{2} - r \right)$. Because

$p, q \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$,

we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$ for every $\epsilon > 0$, where

$$A_1(\epsilon) = \left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - p \right| \geq r + \epsilon \right\}$$

and

$$A_2(\epsilon) = \left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - q \right| \geq r + \epsilon \right\}.$$

Using the properties $F(I)$, we get

$$\left(A_1(\epsilon)^c \cap A_2(\epsilon)^c \right) \in F(I).$$

Thus we write,

$$\begin{aligned} |p - q| &\leq \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - p \right| \\ &\quad + \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - q \right| \\ &< (r + \epsilon) + (r + \epsilon) < 2(r + \epsilon), \end{aligned}$$

for all $(m, n, k) \in A_1(\epsilon)^c \cap A_2(\epsilon)^c$ which is a contradiction. Hence $\text{diam} \left(LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right) \leq 2r$.

Now, consider a arithmetic summable triple sequence

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ such that

$I - \lim_{mnk \rightarrow \infty} \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = l$.

Let $\epsilon > 0$. Then we can write

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq \epsilon \right\} \in I$$

Thus, we have

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - p \right| \leq$$

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| + |l - p| \leq$$

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| + r \leq r + \epsilon,$$
 for each $p \in \bar{B}_r(l) := \{p \in \mathbb{R}^3 : |p - l| \leq r\}$. Then, we get

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - p \right| < r + \epsilon$$

for each

$$(m, n, k) \in \left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| < \epsilon \right\}.$$

Because the arithmetic summable triple sequence

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is I -convergent to l , we have

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| < \epsilon \right\} \in F(I).$$

Therefore, we get

$$p \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right).$$

Consequently, we can write

$$I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) = \bar{B}_r(l). \quad (1)$$

Because $diam(\bar{B}_r(l)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set

$$I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

is not lower bound. ■

Theorem 3.2 *Let I be an admissible ideal. A arithmetic summable triple sequence $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is I -analytic if and only if there exists a non-negative real number r such that $I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi$, for all $r > 0$, an I -analytic arithmetic summable triple sequence always contains a*

sub sequence $\left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right)$ with $I-LIM^{(x_{m_i n_j k_\ell})^r}$ $\left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right) \neq \phi$.

Proof. Because the arithmetic summable triple sequence $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is I -analytic then there exists a positive real number M such that

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right|^{1/m+n+k} \geq M \right\} \in I.$$

Define

$$r' = \sup \left\{ \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right|^{1/m+n+k} \geq M : (m, n, k) \in K^c \right\},$$

where

$$K = \left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right|^{1/m+n+k} \geq M \right\}.$$

Then the set $I-LIM^{r'} \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ contains the origin of \mathbb{R}^3 . So we have

$$I-LIM^{r'} \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi.$$

If $I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \neq \phi$ for some $r \geq 0$, then there exists l such that

$$L \in I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), \text{ i.e.,}$$

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right|^{1/m+n+k} \geq r + \epsilon \right\} \in I,$$

for each $\epsilon > 0$. Then we say that almost all

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ are contained in some ball with any radius greater than r . So the arithmetic summable triple sequence spaces

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is I - analytic.

As $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is a I - analytic arithmetic summable triple sequence in a three-dimensional metric space, it certainly contains a I - convergent sub sequence

$$\left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right).$$

Let l be its I - limit point, then

$$I-LIM^r \left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right) = \bar{B}_r(l)$$

and, for $r > 0$,

$$I-LIM^{(x_{m_i n_j k_\ell})^r} \left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right) \neq \phi.$$

■

Theorem 3.3 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal. If $\left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right)$ is a sub sequence of $\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, then

$$I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \subseteq I-LIM^r \left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right).$$

Proof. The proof is trivial (See [9], Proposition 2.3). ■

Theorem 3.4 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal. The rough I - limit set of an arithmetic summable triple sequence $x = \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), r \right)$ is closed.

Proof. The result is true for

$$I-LIM^r \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), r \right) = \phi.$$

Assume that

$$I-LIM^r \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), r \right) \neq \phi.$$

Then, we can choose a arithmetic summable triple sequence

$\left(\left(\sum_{m_i|u} \sum_{n_j|v} \sum_{k_\ell|w} x_{mnk} - \sum_{m_i|\langle a,u \rangle} \sum_{n_j|\langle b,v \rangle} \sum_{k_\ell|\langle c,w \rangle} x_{mnk} \right), r \right) \subseteq$
 $I-LIM^r \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), r \right)$ such
 that $\left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right), r \right) \rightarrow^r l$ for
 $m, n, k \rightarrow \infty$. To prove

$$l \in I-LIM^r \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), r \right).$$

Let $\epsilon > 0$ be given. Because $\rightarrow l, \exists i, j, \ell = i_{\frac{\epsilon}{2}}, j_{\frac{\epsilon}{2}}, \ell_{\frac{\epsilon}{2}} \in \mathbb{N}^3$ such that

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) - l \right| < \frac{\epsilon}{2}, \forall m \geq i_{\frac{\epsilon}{2}}, n \geq j_{\frac{\epsilon}{2}}, k \geq \ell_{\frac{\epsilon}{2}}.$$

Now choose an $m_0 n_0 k_0 \in \mathbb{N}^3$ such that $m_0 \geq i_{\frac{\epsilon}{2}}, n_0 \geq j_{\frac{\epsilon}{2}}, k_0 \geq \ell_{\frac{\epsilon}{2}}$.

Then we can write

$$|y_{m_0 n_0 k_0} - l| < \frac{\epsilon}{2}.$$

On the other hand, because

$$\begin{aligned} & \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) \\ & \subseteq I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), \end{aligned}$$

we have $y_{m_0 n_0 k_0} \in I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$,
 namely,

$$\begin{aligned} A \left(\frac{\epsilon}{2} \right) = & \left\{ (m, n, k) \in \mathbb{N}^3 : \right. \\ & \left. \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_{m_0 n_0 k_0} \right| \geq r + \frac{\epsilon}{2} \right\} \in I. \end{aligned} \quad (2)$$

Now let us prove that the inclusion

$$A^c \left(\frac{\epsilon}{2} \right) \subseteq A^c(\epsilon) \quad (3)$$

holds, where

$$A(\epsilon) = \left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq r + \epsilon \right\}.$$

Take $(u, v, w) \in A^c\left(\frac{\epsilon}{2}\right)$. Then we have

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_{m_0 n_0 k_0} \right| < r + \frac{\epsilon}{2}$$

and hence

$$\begin{aligned} & \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \\ & \leq \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_{m_0 n_0 k_0} \right| + |y_{m_0 n_0 k_0} - l| < r + \epsilon, \end{aligned}$$

i.e., $(u, v, w) \in A^c(\epsilon)$, which proves (3.3), we get $A(\epsilon) \in I$

(i.e., $l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$).

■

Theorem 3.5 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal. The rough I - limit set of arithmetic summable triple sequence

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is convex.

Proof. Let

$$y_1, y_2 \in l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$$

for arithmetic summable triple sequence

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ and let $\epsilon > 0$ be given. Define

$$A_1(\epsilon) = \left\{ (m, n, k) : \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - y_1 \right| \geq r + \epsilon \right\}$$

and

$$A_2(\epsilon) = \left\{ (m, n, k) : \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - y_2 \right| \geq r + \epsilon \right\}.$$

Because $y_1, y_2 \in I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, we have $A_1(\epsilon), A_2(\epsilon) \in I$. Thus we have

$$\begin{aligned} & \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - [(1-\lambda)y_1 + \lambda y_2] \right| \\ &= | (1-\lambda) \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_1 \right) + \\ & \quad \lambda \left(\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_2 \right) | \\ &\leq (1-\lambda) \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_1 \right| + \\ & \quad \lambda \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y_2 \right| \\ &< (1-\lambda)(r+\epsilon) + \lambda(r+\epsilon) < r+\epsilon \end{aligned}$$

for each $(m, n, k) \in A_1^c(\epsilon) \cap A_2^c(\epsilon)$ and each $\lambda \in [0, 1]$. Because $(A_1^c(\epsilon) \cap A_2^c(\epsilon)) \in F(I)$ by definition of $F(I)$, we get

$$\left\{ \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - [(1-\lambda)y_1 + \lambda y_2] \right| \geq r + \epsilon \right\} \in I,$$

that is

$$[(1-\lambda)y_1 + \lambda y_2] \in I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right),$$

which proves the convexity of the set

$$I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right). \blacksquare$$

Theorem 3.6 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal. Suppose $r > 0$, then a arithmetic summable triple sequence

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk}\right)$ is rough I - convergent to l if and only if there exists a arithmetic summable triple sequence

$y = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk}\right)$ such that

$I - \lim \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk}\right) = l$ and

$\left|\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk}\right)\right.$

$\left. - \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk}\right)\right| \leq r, \dots \dots \dots *$

** for each $m, n, k \in \mathbb{N}$.

Proof. Assume that $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk}\right)$ is rough I - convergent to l . Then we have

$$I - \limsup \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \leq r. \quad (4)$$

Now, define

$$y_{mnk} = \begin{cases} l, & \text{if } \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \leq r, \\ \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) + r \\ \left(\frac{l - \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk}}{\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right|} \right), & \text{otherwise,} \end{cases}$$

Then, we write

$$\begin{aligned} & \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} - l \right| \\ &= \begin{cases} |l - l|, & \text{if } \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \leq r, \\ \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| + r \\ \left(\frac{|l - l| - \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right|}{\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right|} \right), & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned}
& (i.e) \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) - l \right| \\
&= \begin{cases} 0, & \text{if } \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \leq r, \\ \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| - \\ r \left(\frac{\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right|}{\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right|} \right), & \text{otherwise,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
& (i.e) \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right| - l \\
&= \begin{cases} 0, & \text{if } \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \leq r, \\ \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| - r, & \text{otherwise.} \end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
& \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \geq \\
& \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| - r \implies \\
& \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right. \\
& \left. - \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) + l \right| \leq r
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right. \\
& \left. - \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right| \leq r \quad (5)
\end{aligned}$$

for all $m, n, k \in \mathbb{N}$. By equation (3.4) and by definition of

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right)$, we get

$$I - \limsup \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} - l \right| = 0.$$

$$\implies I - \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) \rightarrow^r l.$$

Assume that $(***)$ holds. Because

I - $\lim \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) = l$, we have

$$A(\epsilon) = \left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) - l \right| \geq r + \epsilon \right\} \in I,$$

for each $\epsilon > 0$. Now, define the set

$$B(\epsilon) = \left\{ (m, n, k) \in \mathbb{N}^3 : \left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - l \right| \geq r + \epsilon \right\} \in I.$$

We have

$$B(\epsilon) \subseteq A(\epsilon)$$

holds. Since $A(\epsilon) \in I \implies B(\epsilon) \in I$. Hence,

$x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is rough I -convergent to l . ■

Note 3.7 If we replace the condition

$$\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} - \sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} + \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right| \leq r$$

for all $(m, n, k) \in \mathbb{N}^3$ in the hypothesis of the above theorem with the condition

$$\left\{ \begin{array}{l} \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right. \\ \left. - \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) \right| > r \end{array} \right\} \in I$$

then the theorem will also be valid.

Theorem 3.8 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal. For an arbitrary $c \in I \Gamma \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ of arithmetic summable triple sequence

$x = \left\{ \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), |l - c| \leq r \right.$ for all $l \in I - LIM^r \left. \left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \right| \right\} \right\}$.

Proof. Assume on the contrary that there exist a point $c \in I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right)$ and $l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ such that $|l - c| > r$. Define $\epsilon := \frac{|l-c|-r}{3}$. Then

$$\{|l - c| < \epsilon\} \subseteq \left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq r + \epsilon \right\}. \quad (6)$$

Since $c \in I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right)$, we have

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - c \right| < \epsilon \right\} \notin I.$$

But from definition of I -convergence, since

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq r + \epsilon \right\} \in I,$$

so by (3.6) we have

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - c \right| < \epsilon \right\} \in I,$$

which contradicts the fact $c \in I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right)$.

On the other hand, if $c \in I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right)$ i.e.,

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - c \right| < \epsilon \right\} \notin I,$$

then

$$\left\{ (m, n, k) \in \mathbb{N}^3 : \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq r + \epsilon \right\} \notin I,$$

which contradicts the fact

$$l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right).$$

■

Theorem 3.9 Let $I \subset 3^{\mathbb{N}}$ be an arithmetic summable admissible ideal.

- (i) If $c \in I \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right)$ then
 $I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \subseteq \bar{B}_r(c)$.
- (ii) $I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) =$
 $\bigcap_{c \in I} \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right) \bar{B}_r(c) =$
 $\left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right) \subseteq \bar{B}_r(l) \right\}.$

Proof. (i) If $c \in I \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right)$ then by Theorem (3.8), we have $|l - c| \leq r$ for all $l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, other wise we get

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - l \right| \geq r + \epsilon \right\} \notin I,$$

for $\epsilon := \frac{|l-c|-r}{3}$. Because c is an I - cluster point of

$\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$, this contradicts with the fact that $l \in I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$.

(ii) From (i) we have

$$I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \subseteq \bigcap_{c \in I} \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right) \bar{B}_r(c) \dots \dots (A)$$

Now, let $y \in \bigcap_{c \in I} \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right) \bar{B}_r(c)$. Then

we have

$$\left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} y_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} y_{mnk} \right) - c \right| \leq r,$$

for all $c \in I \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right)$, which is equivalent to $I \left(\Gamma_{(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk})} \right) \subseteq \bar{B}_r(y)$, i.e.,

$$\bigcap_{c \in I} \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \bar{B}_r(c) = \\ \left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \bar{B}_r(l) \right\} \cdots (B)$$

Now, let $y \notin I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$. Then, there exists an $\epsilon > 0$ such that

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - y \right| \geq r + \epsilon \right\} \notin I, \\ \implies \text{the existence of an } I\text{-cluster point } c \text{ of the sequence} \\ \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \text{ with } |y - c| \geq r + \epsilon, \\ \text{i.e.,}$$

$$I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \not\subseteq \bar{B}_r(y) \text{ and} \\ y \notin \left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \bar{B}_r(l) \right\}. \\ \text{Hence } y \in I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \\ \text{follows from}$$

$$y \in \left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \bar{B}_r(l) \right\},$$

i.e.,

$$\left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \bar{B}_r(l) \right\} \subseteq \\ I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \cdots \cdots (C).$$

Therefore, the inclusions (A)-(B) ensure that (C) holds i.e.,

$$I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) \\ \bigcap_{c \in I} \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \bar{B}_r(c) = \\ \left\{ l \in \mathbb{R}^3 : I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \bar{B}_r(l) \right\}. \blacksquare$$

Theorem 3.10 Let $I \subset \mathbb{Z}^{\mathbb{N}}$ be an arithmetic summable admissible ideal and $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ be an arithmetic summable triple I -analytic sequence, if

$$r \geq \text{diam} \left(I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \right),$$

$$\text{then we have have } I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \subseteq \\ I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right).$$

Proof. Let $c \notin I-LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$.

Then there exists an $\epsilon > 0$ such that

$$\left\{ \left| \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right) - c \right|^{1/m+n+k} \geq r + \epsilon \right\} \notin I. \quad (7)$$

Since $x = \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)$ is I - analytic and from the inequality (3.7), there exists an I -cluster point c_1 such that

$$|c - c_1| > r + \epsilon_1, \text{ where } \epsilon_1 := \frac{\epsilon}{2}.$$

So we get

$diam \left(I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \right) > r + \epsilon_1$. The converse of this theorem is also true, i.e., if

$$\begin{aligned} & I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \\ & \subseteq I - LIM^r \left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right), \end{aligned}$$

then we have $r \geq diam \left(I \left(\Gamma_{\left(\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u \rangle} \sum_{n|\langle b,v \rangle} \sum_{k|\langle c,w \rangle} x_{mnk} \right)} \right) \right)$.

■

4 Open Problem

In this paper we studied some basic properties of arithmetic rough summable of I - convergent for triple sequence and the set of all arithmetic rough summable of I - limits for a triple sequence and relation between analytic ness and arithmetic rough summable of I - statistical convergence for a triple sequences. It is open problem that what are the results of this study in conventional sequence spaces?

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