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Generalized Sauleh Distribution with Statistical Properties and Applications in Biological and Climatical Science

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Abstract

Statistical distributions have wide range of applications in distinct fields of science. Several times standard distributions did not provide an adequate fit of real life data sets. To overcome such difficulties, statisticians introduced various techniques by which one can easily formulate or generalize these distributions. So that the resulting distributions become richer or efficient while analyzing data sets. This paper suggests the generalization of Sauleh distribution (GSD). By using linear convex combination technique, with this technique two more parameters were added to the parent distribution. The formulated distribution becomes more flexible than Sauleh distribution. Detailed description about the properties of a generalized Sauleh distribution including moments, moment generating function, deviations about mean and median, stochastic orderings, Bonferroni and Lorenz curves, Renyi entropy, order statistics, survival function, hazard rate function and mean residual function has been discussed. The behavior of a probability density function (p.d.f) and cumulative distribution function

(c.d.f) have been depicted through graphs. The parameters of the distribution are estimated by the known method of maximum likelihood estimation. The performance of the formulated distribution has been illustrated through two real data sets and we accomplish that the formulated distribution provide better fit than compared competitive distributions.

Keywords: Moments, moment generating function, reliability measures, mean deviations, maximum likelihood function.

Mathematics subject classification: 60-XX, 62-XX, 11-KXX

1 Introduction

The classical distributions of statistics show significant role in many applied science such as bio-medicine, engineering, finance and actuarial science. The real life data obtained during different investigations does not follow the standard distributions for modeling. In such cases statisticians face different types of difficulties, to overcome such difficulties researchers made extensions and generalization to several distribution with addition of extra parameters. For analysis and modeling of real life data, exponential and gamma distribution are popular distributions. Among these two distributions, gamma distribution has scale as well as shape parameters. Because of these parameters it provides more flexibility to analyze real life data. It has increasing as well as decreasing failure rate while exponential distribution, has constant hazard rate and memory less property. Gamma distribution has various advantages, but it also has one major disadvantage. If shape parameter of gamma distributions is not integer, then its cumulative distributions function is incomplete gamma function which cannot be expressed in compact form. In the literature of statistics, various extensions of these distributions have been made for more flexibility. Lindley (1) has proposed one parameter life time distribution having the following probability density function.

$$f(y;\theta)=\frac{\theta^2}{\theta+1}(1+y)e^{-\theta y}\quad ;y>0,\theta>0.$$

In recent years researchers have made a lot of work on Lindley distribution and they have proposed one as well as two parameter distributions, for modeling different complex data. An elaborate study on Lindley distribution has been done by Ghitney et al (2), they have laid out that Lindley distribution provide better results than exponential distribution for the waiting times before the service of the bank customers.

They also illustrates that the contours of hazard rate function of the Lindley distribution is an increasing function while as the mean residual life is a decreasing function of the random variable. Zakerzadeh and Dolati (3), Nadarajah et al (4), they extended Lindley distribution with addition of new parameters and expounded the performance of the extended distribution through data sets. In recent years many authors have made different contributions to modify the Lindley distribution. Merovci (5), has introduced transmuted Lindley distribution and discussed its several properties. Sharma et al has introduced the inverse of the Lindley distribution and propounded its different properties. Shanker et al (6) developed a new life time distribution named Akash distribution which were proved superior then exponential and Lindley distribution. Subhradev sen et al (7) has proposed a one parameter distribution called xgamma distribution and they have proved by an application that xgamma distribution provide better fit than exponential distribution. K.K Shukla (8), have suggested the Pranav distribution and studied its different properties.. Ahmad et al (9) has introduced transmuted inverse Lindley distributions and estimates its properties. Ahmad et al (10) has developed a new distribution and name it Hamza distribution and studied its different properties. Recently Ahmad et al (11) proposed one parameter Sauleh distribution by using the method of linear convex combination of exponential and gamma distribution. Its several mathematical properties have been studied. These distributions have benefits as well as drawbacks over one another to analyze more complex data. In this paper an attempt has been made to establish a generalization of Sauleh distribution which is more pliable and provide better results than the existing ones.

The probability density function of the generalized Sauleh distribution (GSD) follows.

$$f(y;\alpha,\beta,\theta)=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left(\theta+\alpha y^2+\beta y^3\right)e^{-\theta y}\ ; y>0,\alpha,\beta,\theta>0 \qquad (1.1)$$

The generalized Sauleh distribution is a linear convex combination of exponential (θ), gamma $(\theta,3)$ and gamma $(\theta,4)$ distributions.

$$f(y;\alpha,\beta,\theta)=p_1 g_1(y,\theta)+p_2 g_2(y,\theta)+[1-(p_1+p_2)]g_3(y,\theta).$$

Where $$p_1=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\ ;\ p_2=\frac{2\alpha\theta}{\theta^4+2\alpha\theta+6\beta}$$

$$g_1(y,\theta)=\theta e^{-\theta y};y>0,\theta>0 \quad ; \; g_2(y,\theta)=\frac{\theta^3}{2}y^2e^{-\theta y};y>0,\theta>0$$

And $$g_3(y,\theta)=\frac{\theta^4}{6}y^3e^{-\theta y};y>0,\theta>0$$

The corresponding cumulative distribution function (c.d.f) of (1.1), is given by

$$F(y;\alpha,\beta,\theta)=1-\left[1+\frac{\theta y\left(\beta\theta^2y^2+(\alpha\theta+3\beta)\theta y+2(\alpha\theta+3\beta)\right)}{\theta^4+2\alpha\theta+6\beta}\right]e^{-\theta y};y>0;\alpha,\beta,\theta>0 \tag{1.2}$$

The graph of the p.d.f of GSD for different values of parameters is shown in figures (1.1) and (1.2).

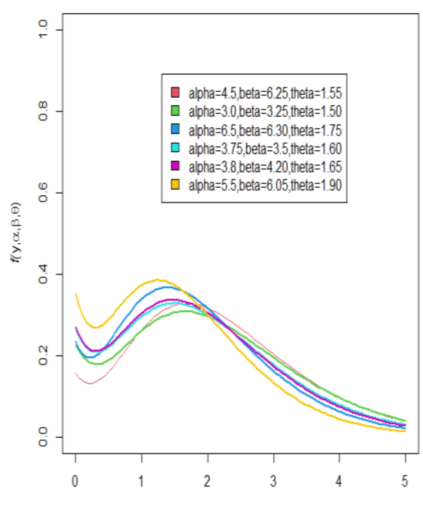


Figure 1.1: pdf of GSD under different values to parameters

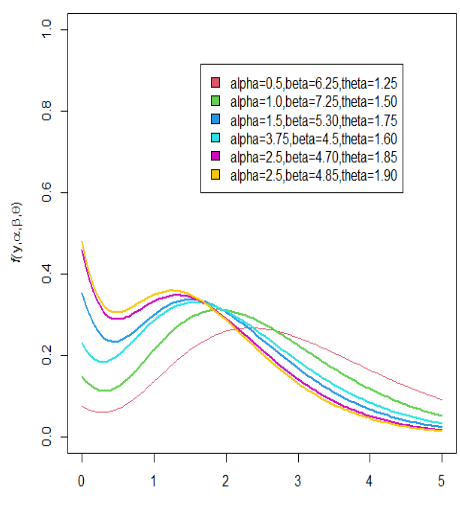


Figure 1.2: pdf of GSD under different values to parameters

The graph of c.d.f of GSD for different values of parameters are shown in figures (2.1) and (2.2)

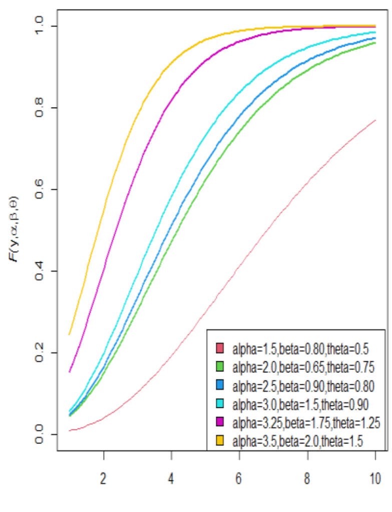


Figure 2.1: cdf of GSD under different values to parameters

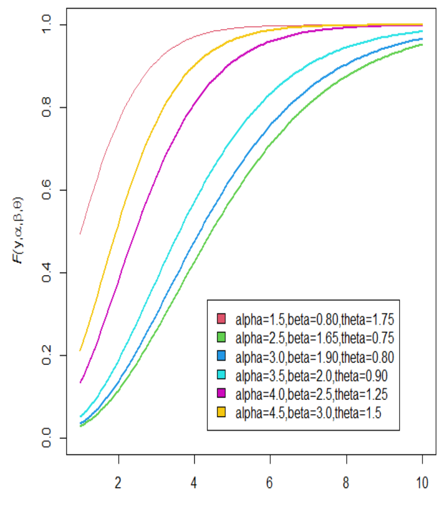


Figure 2.2: cdf of GSD under different values to parameters

Some Special Cases of GSD

1. If $\alpha = \beta = 1$, then equation (1.1) reduces to Sauleh distribution having following probability density function

$$f(y;\theta) = \frac{\theta^4}{\theta^4 + 2\theta + 6}\left(\theta + y^2 + y^3\right)e^{-\theta y}; y > 0, \theta > 0$$

2. If $\alpha = 1, \beta = 0$, then equation (1.1) reduces to Ishita distribution having following probability density function

$$f(y;\theta) = \frac{\theta^3}{\theta^3 + 2}\left(\theta + y^2\right)e^{-\theta y}; y > 0, \theta > 0$$

3. If $\alpha = 0, \beta = 1$, then equation (1.1) reduces to Pranav distribution having following probability density function

$$f(y;\theta) = \frac{\theta^4}{\theta^4 + 6}\left(\theta + y^3\right)e^{-\theta y}; y > 0, \theta > 0$$

4. If $\alpha = \frac{\theta^2}{2}, \beta = 0$, then equation (1.1) reduces to Xgamma distribution having following probability density function

$$f(y;\theta) = \frac{\theta^2}{\theta + 1}\left(1 + \frac{\theta}{2}y^2\right)e^{-\theta y}; y > 0, \theta > 0$$

5. If $\alpha = 0, \beta = 0$, then equation (1.1) reduces to exponential distribution having following Probability density function

$$f(y;\theta) = \theta e^{-\theta y}; y > 0, \theta > 0$$

2 Mathematical Properties of Generalized Sauleh Distribution

2.1 Moments of Generalized Sauleh Distribution

Let Y be a random variable follows generalized Sauleh distribution. Then r^{th} moment denoted by $\mu_r^{'}$ is given as

$$\mu_r^{'} = E(Y^r) = \int_0^\infty y^r f(y, \alpha, \beta, \theta) dy$$

$$= \int_0^\infty \frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta} y^r (\theta + \alpha y^2 + \beta y^3) e^{-\theta y} dy$$

$$= \frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta} \int_0^\infty (\theta y^r + \alpha y^{2+r} + \beta y^{3+r}) e^{-\theta y} dy$$

$$= \frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta} \left[\frac{\theta\Gamma(r+1)}{\theta^{r+1}} + \frac{\alpha\Gamma(r+3)}{\theta^{r+3}} + \frac{\beta\Gamma(r+4)}{\theta^{r+4}} \right]$$

$$= \frac{r!\{\theta^4 + \alpha\theta(r+1)(r+2) + \beta(r+1)(r+2)(r+3)\}}{\theta^r (\theta^4 + 2\alpha\theta + 6\beta)}$$

Now substituting $r = 1,2,3,4$ we obtain first four moments about origin of generalized Sauleh distribution

$$\mu_1^{'} = \frac{\theta^4 + 6\alpha\theta + 24\beta}{\theta(\theta^4 + 2\alpha\theta + 6\beta)} \qquad \mu_2^{'} = \frac{2(\theta^4 + 12\alpha\theta + 60\beta)}{\theta^2(\theta^4 + 2\alpha\theta + 6\beta)}$$

$$\mu_3^{'} = \frac{6(\theta^4 + 20\alpha\theta + 120\beta)}{\theta^3(\theta^4 + 2\alpha\theta + 6\beta)} \qquad \mu_4^{'} = \frac{24(\theta^4 + 30\alpha\theta + 210\beta)}{\theta^4(\theta^4 + 2\alpha\theta + 6\beta)}$$

The moments about mean of the generalized Sauleh distribution are obtained by using relationship between moments about mean and moments about origin

$$\mu_2 = \frac{\theta^8 + 16\alpha\theta^5 + 84\beta\theta^4 + 12\alpha^2\theta^2 + 240\alpha\beta\theta + 144\beta^2}{\theta^2\left(\theta^4 + 2\alpha\theta + 6\beta\right)^2}$$

$$\mu_3 = \frac{\begin{Bmatrix} 2\theta^{12} + 48\alpha\theta^9 + 396\beta\theta^8 + 50\alpha^2\theta^6 + 72(7\alpha - 3)\beta\theta^5 + 1512\beta^2\theta^4 \\ + 912\alpha^3\theta^3 + 432(16\alpha - 7)\alpha\beta\theta^2 + 864(15\alpha - 16)\beta^2\theta + 1728\beta^3 \end{Bmatrix}}{\theta^3\left(\theta^4 + 2\alpha\theta + 6\beta\right)^3}$$

$$\mu_4 = \frac{\begin{Bmatrix} 9\theta^{16} + 108\theta^{13} + 1080\beta\theta^{12} + \left(24\alpha^2 + 120\alpha - 6\right)\theta^{10} + 25520\alpha\beta\theta^9 + 103944\beta^2\theta^8 \\ + \left(5568\alpha^2 + 2256\alpha - 516\right)\alpha\theta^7 + (129040\alpha^2\beta + 18144\alpha\beta - 3024)\theta^6 \\ + (60350\beta - 12384\alpha)\alpha\beta\theta^5 + \left(933120\beta^3 - 14544\alpha^4 - 11520\alpha^3 - 2880\alpha^2\right)\theta^4 \\ + \left(51840\alpha^2 - 23040\beta - 185472\alpha^2\beta - 92160\alpha\beta\right)\alpha\theta^3 \\ - \left(183168\alpha^2 + 276480\alpha + 69120\right)\beta^2\theta^2 + \left(23328\alpha\beta + 324\right)\alpha^2\theta^2 \\ + 1460064\alpha\beta^3\theta - 995328\beta^4 \end{Bmatrix}}{\theta^4\left(\theta^4 + 2\alpha\theta + 6\beta\right)^4}$$

The standard deviation (S.D), coefficient of variation (c.v), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, index of dispersion (γ) of generalized Sauleh distribution are determined as

$$\sigma = \frac{\sqrt{\theta^8 + 16\alpha\theta^5 + 84\beta\theta^4 + 12\alpha^2\theta^2 + 240\alpha\beta\theta + 144\beta^2}}{\theta\left(\theta^4 + 2\alpha\theta + 6\beta\right)}$$

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^8 + 16\alpha\theta^5 + 84\beta\theta^4 + 12\alpha^2\theta^2 + 240\alpha\beta\theta + 144\beta^2}}{\theta^4 + 6\alpha\theta + 24\beta}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}} = \frac{\begin{Bmatrix} 2\theta^{12} + 48\alpha\theta^9 + 396\beta\theta^8 + 50\alpha^2\theta^6 + 72(7\alpha - 3)\beta\theta^5 + 1512\beta^2\theta^4 \\ + 912\alpha^3\theta^3 + 432(16\alpha - 7)\alpha\beta\theta^2 + 864(15\alpha - 16)\beta^2\theta + 1728\beta^3 \end{Bmatrix}}{\left(\theta^8 + 16\alpha\theta^5 + 84\beta\theta^4 + 12\alpha^2\theta^2 + 240\alpha\beta\theta + 144\beta^2\right)^{\frac{3}{2}}}$$

$$\beta_1=\frac{\mu_4}{(\mu_2)^2}=\frac{\left\{\begin{array}{l}9\theta^{16}+108\theta^{13}+1080\beta\theta^{12}+\left(24\alpha^2+120\alpha-6\right)\theta^{10}+25520\alpha\beta\theta^9\\+103944\beta^2\theta^8+\left(5568\alpha^2+2256\alpha-516\right)\alpha\theta^7\\+(129040\alpha^2\beta+18144\alpha\beta-3024)\theta^6+(60350\beta-12384\alpha)\alpha\beta\theta^5\\+\left(933120\beta^3-14544\alpha^4-11520\alpha^3-2880\alpha^2\right)\theta^4\\+\left(51840\alpha^2-23040\beta-185472\alpha^2\beta-92160\alpha\beta\right)\alpha\theta^3\\-\left(183168\alpha^2+276480\alpha+69120\right)\beta^2\theta^2+\left(23328\alpha\beta+324\right)\alpha^2\theta^2\\+1460064\alpha\beta^3\theta-995328\beta^4\end{array}\right\}}{\left(\theta^8+16\alpha\theta^5+84\beta\theta^4+12\alpha^2\theta^2+240\alpha\beta\theta+144\beta^2\right)^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1^{'}}=\frac{\theta^8+16\alpha\theta^5+84\beta\theta^4+12\alpha^2\theta^2+240\alpha\beta\theta+144\beta^2}{\theta\left(\theta^4+2\alpha\theta+6\beta\right)\left(\theta^4+6\alpha\theta+24\beta\right)}$$

2.2 Moment Generating Function of Generalized Sauleh Distribution

Let Y be a random variable follows generalized Sauleh distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$ is given

$$M_Y(t)=E\left(e^{ty}\right)=\int_0^\infty e^{ty}f\left(y;\alpha,\beta,\theta\right)dy$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\int_0^\infty e^{ty}\left(\theta+\alpha y^2+\beta y^3\right)e^{-\theta y}dy$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\int_0^\infty\left(\theta+\alpha y^2+\beta y^3\right)e^{-(\theta-t)y}dy$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left[\frac{\theta}{\theta-t}+\frac{2\alpha}{(\theta-t)^3}+\frac{6\beta}{(\theta-t)^4}\right]$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left[\frac{1}{\left(1-\frac{t}{\theta}\right)}+\frac{2\alpha}{\theta^3\left(1-\frac{t}{\theta}\right)^3}+\frac{6\beta}{\theta^4\left(1-\frac{t}{\theta}\right)^4}\right]$$

$$=\frac{1}{\theta^4+2\alpha\theta+6\beta}\left[\theta^4\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+2\alpha\theta\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k+6\beta\sum_{k=0}^{\infty}\binom{k+3}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$=\sum_{k=0}^{\infty}\frac{\theta^4+\alpha\theta(k+1)(k+2)+\beta(k+1)(k+2)(k+3)}{\theta^4+2\alpha\theta+6\beta}\left(\frac{t}{\theta}\right)^k$$

3 Reliability Measures

Suppose Y be a continuous random variable with c.d.f $F(y), y \geq 0$ .Then its reliability function which is also called survival function is defined as

$$S(y)=p_r(Y>y)=\int_y^{\infty} f(y)dy=1-F(y)$$

Therefore, the survival function for generalized sauleh distribution is given as

$$S(y,\alpha,\beta,\theta)=1-F(y,\alpha,\beta,\theta)$$

$$=\left[1+\frac{\theta y\left(\beta\theta^2y^2+(\alpha\theta+3\beta)\theta y+2(\alpha\theta+3\beta)\right)}{\theta^4+2\alpha\theta+6\beta}\right]e^{-\theta y} \qquad (3.1)$$

The hazard rate function of a random variable y is given as

$$H(y,\alpha,\beta,\theta)=\frac{f(y,\alpha,\beta,\theta)}{S(y,\alpha,\beta,\theta)} \qquad (3.2)$$

Using equation (1.1) and equation (3.1) in (3.2), we get

$$H(y,\alpha,\beta,\theta)=\frac{\theta^4\left(\theta+\alpha y^2+\beta y^3\right)}{\left\{\theta^4+2\alpha\theta+6\beta+\theta y\left(\beta\theta^2y^2+(\alpha\theta+3\beta)+2(\alpha\theta+3\beta)\right)\right\}} \qquad (3.2)$$

The mean residual function denoted by $m(y)$, and is defined as

$$m(y)=E[Y-y/Y>y]=\frac{1}{S(y,\alpha,\beta,\theta)}\int_y^{\infty}1-F(t)dt$$

Therefore, the mean residual function of generalized Sauleh distribution is given by

$$m(y)=\frac{\beta\theta^3y^3+(\alpha\theta+6\beta)\theta^2y^2+2(2\alpha\theta+9\beta)\theta y+\theta^4+6\alpha\theta+24\beta}{\theta\left[\beta\theta^3y^3+(\alpha\theta+3\beta)\theta^2y^2+2(\alpha\theta+3\beta)\theta y+\theta^4+2\alpha\theta+6\beta\right]}$$

We observe that $H(0)=f(0)=\dfrac{\theta^5}{\theta^4+2\alpha\theta+6\beta}$ and $m(0)=\mu_1^{'}=\dfrac{\theta^4+6\alpha\theta+24\beta}{\theta\left(\theta^4+2\alpha\theta+6\beta\right)}$

The graphs of the survival function of GSD for different values of parameters are shown in figures (3.1) and (3.2).

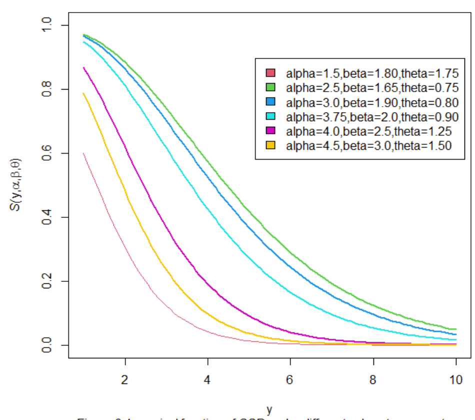


Figure 3.1: survival function of GSD under different values to parameters

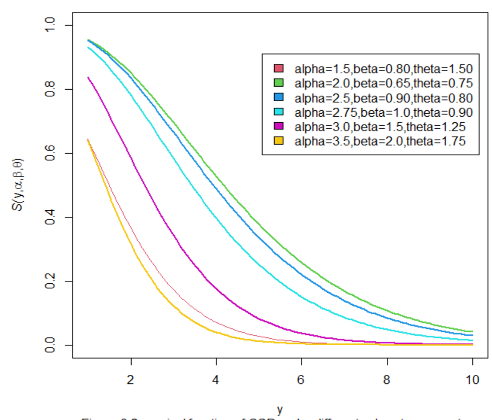


Figure 3.2: survival function of GSD under different values to parameters

The graph of the hazard rate function of GSD for different values of parameters are shown in figures (4.1) and (4.2).

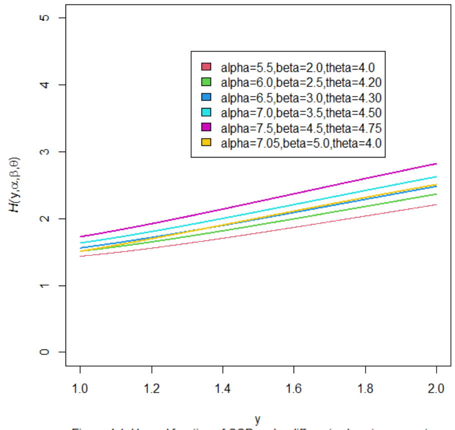


Figure 4.1: Hazard function of GSD under different values to parameters

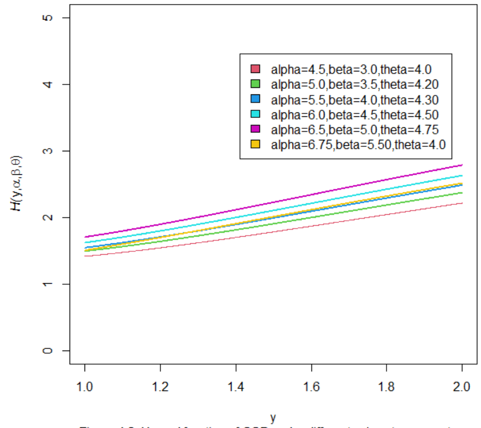


Figure 4.2: Hazard function of GSD under different values to parameters

The graph of the mean residual function of GSD for different values of parameters are shown in figures (5.1) and (5.2)

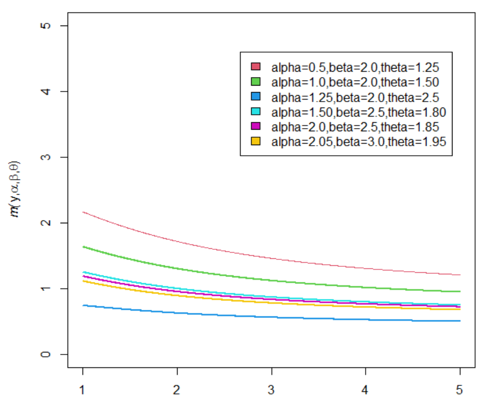


Figure 5.1: mean residual function of GSD under different values to parameters

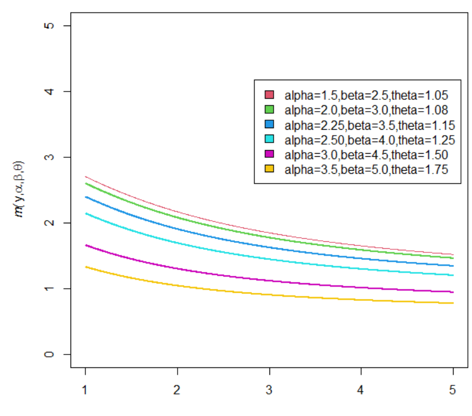


Figure 5.2: mean residual function of GSD under different values to parameters

4 Renyi Entropy

If Y denotes a continuous random variable having probability density function $f(y,\alpha,\beta,\theta)$. Then

Renyi entropy is defined as

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty f^{\delta}(y)dy\right\},\ \text{where}\ \delta>0\ \text{and}\ \delta\neq 1$$

Thus, the Renyi entropy for generalized Sauleh distribution (1.1), is given as

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty\left[\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left(\theta+\alpha y^2+\beta y^3\right)e^{-\theta y}\right]^\delta dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\int_0^\infty\frac{\theta^{4\delta}}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\left(\theta+\alpha y^2+\beta y^3\right)^\delta e^{-\theta\delta y}dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\frac{\theta^{5\delta}}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\int_0^\infty\left(1+\frac{\alpha}{\theta}y^2+\frac{\beta}{\theta}y^3\right)^\delta e^{-\theta\delta y}dy\right\}$$

Using the binomial theorem, $(1+x)^n=\sum_{r=0}^\infty\binom{n}{r}x^r$, we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\theta^{5\delta}}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\int_0^\infty\sum_{r=0}^\infty\binom{\delta}{r}\left(\frac{\alpha}{\theta}y^2+\frac{\beta}{\theta}y^3\right)^r e^{-\theta\delta y}dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\frac{\theta^{5\delta}}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\int_0^\infty\sum_{r=0}^\infty\binom{\delta}{r}\left(\frac{\alpha}{\theta}y^2\right)^r\left(1+\frac{\beta}{\alpha}y\right)^r e^{-\theta\delta y}dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\frac{\theta^{5\delta}}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\sum_{r=0}^\infty\binom{\delta}{r}\int_0^\infty\sum_{q=0}^\infty\binom{r}{q}\left(\frac{\alpha}{\theta}y^2\right)^r\left(\frac{\beta}{\alpha}y\right)^q e^{-\theta\delta y}dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{r=0}^\infty\binom{\delta}{r}\sum_{q=0}^\infty\binom{r}{q}\left[\frac{\theta^{5\delta-r}\alpha^{r-q}\beta^q}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\int_0^\infty y^{2r+q}e^{-\theta\delta y}dy\right]\right\}$$

Using, $\int_0^\infty y^n e^{-ay}=\frac{n!}{a^{n+1}}$, we get

$$=\frac{1}{1-\delta}\log\left\{\sum_{r=0}^\infty\binom{\delta}{r}\sum_{q=0}^\infty\binom{r}{q}\left[\frac{\theta^{5\delta-r}\alpha^{r-q}\beta^q}{\left(\theta^4+2\alpha\theta+6\beta\right)^\delta}\frac{(2r+q)!}{(\theta\delta)^{2r+q+1}}\right]\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{r=0}^{\infty}\binom{\delta}{r}\sum_{q=0}^{\infty}\binom{r}{q}\left[\frac{\theta^{5\delta-3r-q-1}\alpha^{r-q}\beta^{q}}{\left(\theta^{4}+2\alpha\theta+6\beta\right)^{\delta}}\frac{(2r+q)!}{(\delta)^{2r+q+1}}\right]\right\}$$

5 Mean Deviation from Mean of Generalized Sauleh Distribution

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations. Let Y be a random variable from generalized Sauleh distribution with mean μ.Then the mean deviation from mean is defined as.

$$D(\mu)=E(|Y-\mu|)$$

$$=\int_{0}^{\infty}|Y-\mu|f(y)dy$$

$$=\int_{0}^{\mu}(\mu-y)f(y)dy+\int_{\mu}^{\infty}(y-\mu)f(y)dy$$

$$=\mu\int_{0}^{\mu}f(y)dy-\int_{0}^{\mu}yf(y)dy+\int_{\mu}^{\infty}yf(y)dy-\int_{\mu}^{\infty}\mu f(y)dy$$

$$=\mu F(\mu)-\int_{0}^{\mu}yf(y)dy-\mu[1-F(\mu)]+\int_{\mu}^{\infty}yf(y)dy$$

$$=2\mu F(\mu)-2\int_{0}^{\mu}yf(y)dy \tag{5.1}$$

Now

$$\int_{0}^{\mu}yf(y)dy=\int_{0}^{\mu}y\frac{\theta^{4}}{\theta^{4}+2\alpha\theta+6\beta}\left(\theta+\alpha y^{2}+\beta y^{3}\right)e^{-\theta y}dy$$

$$=\frac{\theta^{4}}{\theta^{4}+2\alpha\theta+6\beta}\int_{0}^{\mu}\left(\theta y+\alpha y^{3}+\beta y^{4}\right)e^{-\theta y}dy$$

After solving the integral, we get

$$\int_0^{\mu} yf(y)dy = \mu - \frac{\left\{\begin{array}{l}\beta\theta^4\mu^4 + (\alpha\theta+4\beta)\theta^3\mu^3 + 3(\alpha\theta+4\beta)\theta^2\mu^2 \\ +6(\alpha\theta+4\beta)\theta\mu + \theta^5\mu + \theta^4 + 6\alpha\theta + 24\beta\end{array}\right\} e^{-\theta\mu}}{\theta(\theta^4+2\alpha\theta+6\beta)} \qquad (5.2)$$

Substituting equation (5.2) in equation (5.1), we get

$$D(\mu) = \frac{2\left\{\beta\theta^3\mu^3 + (\alpha\theta+4\beta)\theta^2\mu^2 + 2(\alpha\theta+9\beta)\theta\mu + \theta^4 + 6\alpha\theta + 24\beta\right\} e^{-\theta\mu}}{\theta(\theta^4+2\alpha\theta+6\beta)}$$

6 Mean Deviation from Median of Generalized Sauleh Distribution

Let Y be a random variable from generalized Sauleh distribution with media M .Then the mean deviation from median is defined as.

$$D(M) = E(|Y-M|)$$

$$= \int_0^{\infty} |Y-M| f(y)dy$$

$$= \int_0^{M} (M-y)f(y)dy + \int_M^{\infty} (y-M)f(y)dy$$

$$= MF(M) - \int_0^{M} yf(y)dy - M[1-F(M)] + \int_M^{\infty} yf(y)dy$$

$$= \mu - 2\int_0^{M} yf(y)dy \qquad (6.1)$$

Now

$$\int_0^{M} yf(y)dy = \int_0^{M} y \frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}(\theta+\alpha y^2+\beta y^3)e^{-\theta y}dy$$

$$= \frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\int_0^{M} (\theta y+\alpha y^3+\beta y^4)e^{-\theta y}dy$$

After solving the integral, we get

$$\int_0^M yf(y)dy = \mu - \frac{\begin{Bmatrix} \beta\theta^4 M^4 + (\alpha\theta + 4\beta)\theta^3 M^3 + 3(\alpha\theta + 4\beta)\theta^2 M^2 \\ + 6(\alpha\theta + 4\beta)\theta M + \theta^5 M + \theta^4 + 6\alpha\theta + 24\beta \end{Bmatrix} e^{-\theta M}}{\theta(\theta^4 + 2\alpha\theta + 6\beta)} \tag{6.2}$$

Substituting equation (6.2) in equation (6.1), we get

$$D(M) = \frac{2\begin{Bmatrix} \beta\theta^4 M^4 + (\alpha\theta + 4\beta)\theta^3 M^3 + 3(\alpha\theta + 4\beta)\theta^2 M^2 \\ + 6(\alpha\theta + 4\beta)\theta M + \theta^5 M + \theta^4 + 6\alpha\theta + 24\beta \end{Bmatrix} e^{-\theta M}}{\theta(\theta^4 + 2\alpha\theta + 6\beta)} - \mu$$

7 Bonferroni and Lorenz Curves

In economics the relation between poverty and economy is well studied by using Bonferroni and Lorenz curves. Besides that these curves have been used in different fields such as reliability, insurance and biomedicine.

The Bonferroni curves, $B(s)$ is given as

$$B(s) = \frac{1}{s\mu}\int_0^t yf(y)dy \tag{7.1}$$

Or

$$B(s) = \frac{1}{s\mu}\int_0^s F^{-1}(y)dy$$

And Lorenz curves, $L(s)$ is given as

$$L(s) = \frac{1}{\mu}\int_0^t yf(y)dy \tag{7.2}$$

Or

$$L(s) = \frac{1}{\mu}\int_0^s F^{-1}(y)dy$$

Where $E(X)=\mu$ and $t=F^{-1}(s)$

Now

$$\int_0^t yf(y)dy=\mu-\frac{\left\{\begin{array}{l}\beta\theta^4t^4+(\alpha\theta+4\beta)\theta^3t^3+3(\alpha\theta+4\beta)\theta^2t^2\\+6(\alpha\theta+4\beta)\theta t+\theta^5t+\theta^4+6\alpha\theta+24\beta\end{array}\right\}e^{-\theta t}}{\theta(\theta^4+2\alpha\theta+6\beta)} \tag{7.3}$$

Substituting equation (7.3) in equation (7.1) and (7.2), we get

$$B(s)=\frac{1}{s}\left[1-\frac{\left\{\begin{array}{l}\beta\theta^4t^4+(\alpha\theta+4\beta)\theta^3t^3+3(\alpha\theta+4\beta)\theta^2t^2\\+6(\alpha\theta+4\beta)\theta t+\theta^5t+\theta^4+6\alpha\theta+24\beta\end{array}\right\}e^{-\theta t}}{\theta(\theta^4+2\alpha\theta+6\beta)\mu}\right]$$

And

$$L(s)=\left[1-\frac{\left\{\begin{array}{l}\beta\theta^4t^4+(\alpha\theta+4\beta)\theta^3t^3+3(\alpha\theta+4\beta)\theta^2t^2\\+6(\alpha\theta+4\beta)\theta t+\theta^5t+\theta^4+6\alpha\theta+24\beta\end{array}\right\}e^{-\theta t}}{\theta(\theta^4+2\alpha\theta+6\beta)\mu}\right]$$

8 Stochastic Ordering

Stochastic ordering is a rule by which positive continuous random variables are quantified for their comparative performance. A random variable Y is said be smaller than a random variable Z in the

1. Stochastic order $(Y\leq_{st} Z)$ if $F_y(y)\geq F_z(y)$ for all y
2. Hazard rate order $(Y\leq_{ht} Z)$ if $h_y(y)\geq h_z(y)$ for all y
3. Mean residual life order $(Y\leq_{mrl} Z)$ if $m_y(y)\leq m_z(y)$ for all y
4. Likelihood ratio order $(Y\leq_{rl} Z)$ if $\dfrac{f_y(y)}{f_z(y)}$ decreases in y

The below result due to shaked and shanthikumar (12), are famous for demonstrating stochastic ordering of distributions

$$Y \leq_{lr} Z \Rightarrow Y \leq_{hr} Z \Rightarrow Y \leq_{mrl} Z$$

$$\Downarrow Y \leq_{st} Z$$

The following theorem shows that generalized Sauleh distribution is ordered with respect to the strongest 'likelihood ratio ordering'

Theorem: Suppose $Y \sim GSD(\alpha_1, \beta_1, \theta_1)$ and $Z \sim GSD(\alpha_2, \beta_2, \theta_2)$. If

1. $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$
2. $\alpha_1 = \alpha_2, \beta_1 > \beta_2$ and $\theta_1 = \theta_2$
3. $\alpha_1 > \alpha_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$

Then $Y \leq_{lr} Z$ and hence $Y \leq_{hr} Z, Y \leq_{mrl} Z$ and $Y \leq_{st} Z$

Proof: we have

$$\frac{f_Y(y)}{f_Z(y)} = \frac{\theta_1^4\left(\theta_2^4 + 2\theta_2\alpha_2 + 6\beta_2\right)}{\theta_2^4\left(\theta_1^4 + 2\theta_1\alpha_1 + 6\beta_1\right)}\left[\frac{\theta_1 + \alpha_1 y^2 + \beta_1 y^3}{\theta_2 + \alpha_2 y^2 + \beta_2 y^3}\right]e^{-(\theta_1 - \theta_2)y}; y > 0$$

Applying log, we have

$$\log\frac{f_Y(y)}{f_z(y)} = \log\left[\frac{\theta_1^4\left(\theta_2^4 + 2\theta_2\alpha_2 + 6\beta_2\right)}{\theta_2^4\left(\theta_1^4 + 2\theta_1\alpha_1 + 6\beta_1\right)}\right] + \log\left[\frac{\theta_1 + \alpha_1 y^2 + \beta_1 y^3}{\theta_2 + \alpha_2 y^2 + \beta_2 y^3}\right] - (\theta_1 - \theta_2)y$$

Differentiate w. r. t y, we get

$$\frac{d}{dy}\log\frac{f_Y(y)}{f_z(y)} = \frac{2(\alpha_1\theta_2 - \theta_1\alpha_2)y + 3(\beta_1\theta_2 - \beta_2\theta_1)y^2 + (\alpha_2\beta_1 - \alpha_1\beta_2)y^3}{\left(\theta_1 + \alpha_1 y^2 + \beta_1 y^3\right)\left(\theta_2 + \alpha_2 y^2 + \beta_2 y^3\right)}$$

From conditions $(1)-(3)$, we observe that $\frac{d}{dy}\log\frac{f_Y(y)}{f_z(y)} < 0$, which implies that $Y \leq_{lr} Z$ and hence $Y \leq_{hr} Z, Y \leq_{mrl} Z$ and $Y \leq_{st} Z$

9 Order Statistics

Consider $Y_1, Y_2, ...Y_n$ be random samples from generalized Sauleh distribution. Let $Y_{(1)} < Y_{(2)} < ... < Y_{(n)}$ be the corresponding order statistics. Then the p.d.f of the r^{th} order statistics of the generalized Sauleh distribution, say $X = Y_{(r)}$

$$f_X(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x)[1-F(x)]^{n-r} f(x)$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{k=0}^{\infty} \binom{n-r}{k} (-1)^k F^{r+k-1}(x) f(x)$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{k=0}^{\infty} \binom{n-r}{k} (-1)^k \left\{ 1 - \left[1 + \frac{\theta y\left(\beta\theta^2 y^2 + (\alpha\theta + 3\beta)\theta y + 2(\alpha\theta + 3\beta)\right)}{\theta^4 + 2\alpha\theta + 6\beta} \right] e^{-\theta y} \right\}^{r+k-1}$$
$$\frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta} \left(\theta + \alpha y^2 + \beta y^3\right) e^{-\theta y}$$

The corresponding c.d.f of X is given by

$$F_X(x) = \sum_{j=r}^{n} F^j(x)[1-F(x)]^{n-j}$$

$$F_X(x) = \sum_{j=r}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} (-1)^k F^{j+k}(x)$$

$$= \sum_{j=r}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} (-1)^k \left\{ 1 - \left[1 + \frac{\theta y\left(\beta\theta^2 y^2 + (\alpha\theta + 3\beta)\theta y + 2(\alpha\theta + 3\beta)\right)}{\theta^4 + 2\alpha\theta + 6\beta} \right] e^{-\theta y} \right\}^{j+k}$$

10 Maximum Likelihood Estimation

Let $Y_1, Y_2, .., Y_n$ be a random sample of size n from generalized Sauleh distribution then its likelihood function is given by

$$l=\prod_{i=1}^{n} f\left(y_i,\alpha,\beta,\theta\right)$$

$$=\prod_{i=1}^{n}\frac{\theta^4}{\left(\theta^4+2\alpha\theta+6\beta\right)}\left(\theta+\alpha y_i^2+\beta y_i^3\right)e^{-\theta\sum_{i=1}^{n} y_i}$$

$$=\frac{\theta^{4n}}{\left(\theta^4+2\alpha\theta+6\beta\right)^n}\prod_{i==1}^{n}\left(\theta+\alpha y_i^2+\beta y_i^3\right)e^{-\theta\sum_{i=1}^{n} y_i}$$

Log likelihood function is given by

$$\log l=4n\log\theta-n\log\left(\theta^4+2\alpha\theta+6\beta\right)+\sum_{i=1}^{n}\log\left(\theta+\alpha y_i^2+\beta y_i^3\right)-\theta\sum_{i=1}^{n} y_i \qquad (10.1)$$

Differentiating (10.1) w.r.t the parameters, we get

$$\frac{\partial\log l}{\partial\theta}=\frac{4n}{\theta}-\frac{n\left(4\theta^3+2\alpha\right)}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{1}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)}-\sum_{i=1}^{n} y_i \qquad (10.2)$$

$$\frac{\partial\log l}{\partial\alpha}=\frac{-2n\theta}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{y_i^2}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)} \qquad (10.3)$$

$$\frac{\partial\log l}{\partial\beta}=\frac{-6n}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{y_i^3}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)} \qquad (10.4)$$

Equating equations (10.2), (10.3) and (10.4) to zero, we get

$$\frac{4n}{\theta}-\frac{n\left(4\theta^3+2\alpha\right)}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{1}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)}-n\bar{y}=0 \qquad (10.5)$$

$$\frac{-2n\theta}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{y_i^2}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)}=0 \qquad (10.6)$$

$$\frac{-6n}{\left(\theta^4+2\alpha\theta+6\beta\right)}+\sum_{i=1}^{n}\frac{y_i^3}{\left(\theta+\alpha y_i^2+\beta y_i^3\right)}=0 \qquad (10.7)$$

The equations (10.5), (10.6) and (10.7) are nonlinear equations, cannot be solved explicitly for α,β and θ.Applying the iterative methods such as Newton –Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\gamma}\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ of $\gamma(\alpha,\beta,\theta)$ can be obtained by using the above methods.

Since the MLE of $\hat{\gamma}$ follows asymptotically normal distribution which is given as

$$\sqrt{n}(\hat{\gamma}-\gamma)\rightarrow N\left(0,I^{-1}(\gamma)\right)$$

Where $I^{-1}(\gamma)$ is the limiting variance-covariance matrix of $\hat{\gamma}$ and $I^{-1}(\gamma)$ is a 3×3 Fisher information matrix
i,e

$$I^{-1}(\gamma)=-\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2\log l}{\partial\theta^2}\right) & E\left(\frac{\partial^2\log l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2\log l}{\partial\theta\partial\beta}\right) \\ E\left(\frac{\partial^2\log l}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2\log l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2\log l}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2\log l}{\partial\beta\partial\theta}\right) & E\left(\frac{\partial^2\log l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2\log l}{\partial\beta^2}\right) \end{bmatrix}$$

Where

$$\frac{\partial^2\log l}{\partial\theta^2}=\frac{4n\left(7\theta^8-12\alpha\theta^5-84\beta\theta^4-24\alpha\beta\theta-36\beta^2\right)}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{1}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

$$\frac{\partial^2\log l}{\partial\alpha^2}=\frac{4n\theta^2}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{y_i^{\,4}}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

$$\frac{\partial^2\log l}{\partial\beta^2}=\frac{36n}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{y_i^{\,6}}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

$$\frac{\partial^2\log l}{\partial\theta\partial\alpha}=\frac{\partial^2\log l}{\partial\alpha\partial\theta}=\frac{6n\left(\theta^4-2\beta\right)}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{y_i^{\,2}}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

$$\frac{\partial^2\log l}{\partial\theta\partial\beta}=\frac{\partial^2\log l}{\partial\beta\partial\theta}=\frac{12n\left(2\theta^3+\alpha\right)}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{y_i^{\,3}}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

$$\frac{\partial^2\log l}{\partial\alpha\partial\beta}=\frac{\partial^2\log l}{\partial\beta\partial\alpha}=\frac{12n\theta}{\left(\theta^4+2\alpha\theta+\beta\right)^2}-\sum_{i=1}^{n}\frac{y_i^{\,5}}{\left(\theta+\alpha y_i^{\,2}+\beta y_i^{\,3}\right)^2}$$

Hence the approximate $100(1-\psi)\%$ confidence interval for α, β and θ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\gamma})} \ , \hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\gamma})} \text{ And } \hat{\theta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\gamma})}$$

Where $Z_{\frac{\psi}{2}}$ denotes the ψ^{th} percentile of the standard distribution.

11 Data Analysis

In this section we provide applications which explain the efficiency of the generalized Sauleh distribution (GSD). The formulated distribution is compared with Sauleh distribution (SD), Ishita distribution (ISD), Pranav distribution (PD), Xgamma distribution (XGD), Lindley distribution (LD) and exponential distribution (ED).

In order to compare the distribution models, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information criterion).

$$AIC = 2k - 2\ln l \, ; CAIC = \frac{2kn}{n-k-1} - 2\ln l$$
$$BIC = k\ln n - 2\ln l \text{ And } HQIC = 2k\ln(\ln(n)) - 2\ln l$$

Data 1: The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (13). The data are as follows:0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 1: Descriptive statistics of data set first

Min	Q_1	Median	Mean	Q_3	Skew	Kurt.	Max
0.100	1.077	1.450	1.754	2.240	1.328	4.913	5.550

Table 2: The MLEs and the performance rating of the selected models for data set first

Model	Estimates	$-2\log l$	AIC	AICC	HQIC	BIC
GSD	$\hat{\alpha}=86.215$ $\hat{\beta}=75.502$ $\hat{\theta}=2.0050$	188.50	194.50	194.85	197.22	201.33
SD	$\hat{\theta}=1.4975$	214.74	216.74	216.80	217.65	219.02
ISD	$\hat{\theta}=1.1664$	216.06	218.06	218.12	218.97	220.34
PD	$\hat{\theta}=1.4780$	225.07	227.07	227.12	227.97	229.34
XGD	$\hat{\theta}=1.1495$	220.53	222.53	222.59	223.44	224.81
LD	$\hat{\theta}=0.8744$	213.05	215.05	215.10	215.95	217.32
ED	$\hat{\theta}=0.5702$	224.89	226.89	226.95	227.79	229.16

Data 2: The following data consists of thirty successive March precipitation (in inches), the observation studied by Hinkley (14).

0.77,1.74,0.81,1.20,1.95,1.20,0.47,1.43,3.37,2.20,3.00,3.09,1.51,2.10,0.55,1.62,1.31, 0.32,0.5, 0.81,2.81,1.87,1.18,1.35,4.75,2.48,0.96,1.89,0.90,2.05

Table 3: Descriptive statistics of data set second

Min	Q_1	Median	Mean	Q_3	Skew	Kurt.	Max
0.320	0.915	1.470	1.676	2.087	1.091	4.215	4.750

Table 4: The MLEs and the performance rating of the selected models for data set second

Model	Estimates	$-2\log l$	AIC	AICC	HQIC	BIC
GSD	$\hat{\alpha}=1.4075$ $\hat{\beta}=8.8111$ $\hat{\theta}=2.0732$	76.23	82.23	83.15	83.58	86.44
SD	$\hat{\theta}=1.5360$	87.23	89.23	89.37	89.68	90.63

ISD	$\hat{\theta}=1.1995$	87.73	89.73	89.87	90.18	91.13
PD	$\hat{\theta}=1.5115$	91.04	93.04	93.18	93.48	94.44
XGD	$\hat{\theta}=1.1895$	89.18	91.18	91.32	91.63	92.58
LD	$\hat{\theta}=0.9091$	86.30	88.30	88.44	88.75	89.70
ED	$\hat{\theta}=0.5966$	90.98	92.98	93.12	93.43	94.38

From Table 2 and 4, it has been observed that GSD have the lesser $-2\log l$, AIC, AICC, HQIC and BIC values as compared to SD, ISD, PD, XGD, LD and ED. Hence we can conclude that formulated distribution leads to a better fit as compared ones.

Fig.11.1: Estimated density of GSD for data set 1

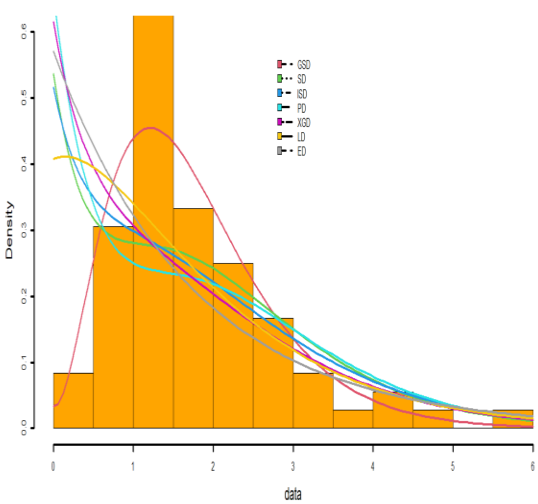
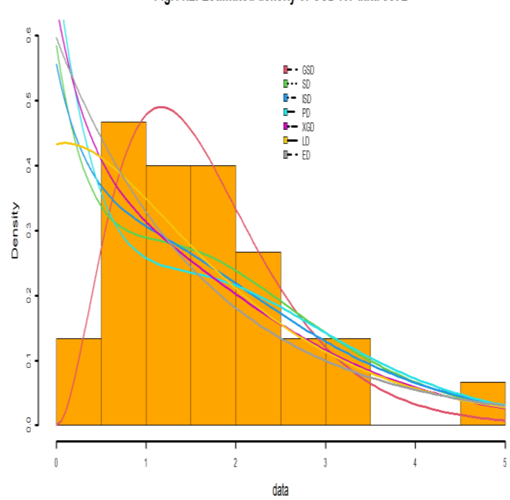


Fig.11.2: Estimated density of GSD for data set 2



12 Concluding remarks

In the present paper a generalized Sauleh distribution has been established. Its several properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, moment generating function, survival function, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy have been discussed. The parameters of the distribution have been estimated by known method of maximum likelihood estimator. Finally the performance of the model has been examined through two data sets and compared with Sauleh, Ishita, Pranav, xgamma, Lindley and exponential distributions. Result shows that generalized Sauleh distribution gives an adequate fit for the data sets.

13 Open Problem

The following probability density function (pdf) represents Power Sauleh distribution.

$$f(y;\alpha,\theta)=\frac{\alpha\theta^4}{\theta^4+2\theta+6}y^{\alpha-1}\left(\theta+y^{2\alpha}+\beta y^{3\alpha}\right)e^{-\theta y^{\alpha}}\;;y>0,\alpha,\theta>0\,.$$

Show that power Sauleh distribution provides better fit than Generalized Sauleh distribution.

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