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Based on a family of bi-univalent functions introduced through the q -analogue of Noor integral operator

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Abstract

Recently, q-analogue of Noor integral operator and other special operators became importance in the field of Geometric Function Theory. In this study, by connecting this operators we introduced an interesting class of bi-univalent functions and obtained coefficient estimates for this new class. Also we present some relevant corollaries.

Keywords: *analytic function, univalent and bi-univalent functions, subordination, coefficient bounds, integral operator.*

1 Introduction

Let $\mathcal{A}$ indicates the family of analytic functions having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\mathfrak{D} = \{z : |z| < 1, z \in \mathbb{C}\}$ and let $S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathfrak{D}\}$.

According the Koebe one-quarter theorem [14], the image of $\mathfrak{D}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. That is, every such univalent function has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{D})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (2)$$

If f and f^{-1} are univalent, then we say that f is bi-univalent function in $\mathfrak{D}$. The class of bi-univalent functions defined in $\mathfrak{D}$ is symbolized by Σ.

One can see important examples in the class in [27]. Although the functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}.\right)$ are in Σ, well known Koebe function is not in Σ. For example, $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are in $\mathcal{S}$ but not in Σ [27].

Given f , $g \in \mathcal{A}$, f is said to be subordinate to g, symbolized

$$f(z) \prec g(z), \quad (3)$$

such that there is an analytic function w defined on $\mathfrak{D}$ with

$$w(0) = 0 \ \text{ and } \ |w(z)| < 1$$

fulfilling the following condition:

$$f(z) = g\left(w(z)\right).$$

The aforecited subclasses of Σ were constructed and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in theTaylor-Maclaurin series expansion (1) were found in several recent studies (see [11, 12] , [15], [27] , [28, 29]) . In very nearly, they have been followed by many works (see also [1], [8], [10, 13, 15, 16, 17], [19], [23] , [22])

Now, we give some basic definitions.

Definition 1.1. *For $q \in (0,1)$, the q-derivative of function $f \in \mathcal{A}$ is defined by (see [18])*

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \quad (4)$$

and

$$\partial_q f(0) = f'(0).$$

Thus we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k,q]\, a_k z^{k-1} \quad (5)$$

where $[k,q]$ is given by

$$[k,q]=\frac{1-q^k}{1-q},\qquad [0,q]=0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]!=\begin{cases}\prod\limits_{n=1}^{k}[n,q], & k\in\mathbb{N}\\ 1, & k=0\end{cases}.$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p}\geq 0$ is given by

$$[\mathfrak{p},q]_k=\begin{cases}\prod\limits_{n=1}^{k}[\mathfrak{p}+n-1,q], & k\in\mathbb{N}\\ 1, & k=0\end{cases}.$$

As $q\to 1$, then we get $[k,q]\to k$. Thus, by choosing the function $g(z)=z^k$, while $q\to 1$, then we obtain

$$\partial_q g(z)=\partial_q z^k=[k,q]\,z^{k-1}=g'(z),$$

where g' is the ordinary derivative.

Recently, function $F^{-1}_{q,\mu+1}(z)$ is defined by Arif et al. [9] by

$$F^{-1}_{q,\mu+1}(z)*F_{q,\mu+1}(z)=z\partial_q f(z),\quad (\mu>-1)$$

where

$$F_{q,\mu+1}(z)=z+\sum_{k=2}^{\infty}\frac{[\mu+1,q]_{k-1}}{[k-1,q]!}z^k,\quad z\in\mathcal{D}. \tag{7}$$

To the series defined in (7) is convergent absolutely in $\mathfrak{D}$, by using the definition of q-derivative through convolution, let us explaine the integral operator $\zeta^{\mu}_q:\mathfrak{D}\to\mathfrak{D}$ by

$$\zeta^{\mu}_q f(z)=F^{-1}_{q,\mu+1}(z)*f(z)=z+\sum_{k=2}^{\infty}\phi_{k-1}a_k z^k,\quad (z\in\mathfrak{D}) \tag{8}$$

where

$$\phi_{k-1}=\frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{9}$$

From (8), one can readily have the identity

$$[\mu+1,q]\zeta^{\mu}_q f(z)=[\mu,q]\zeta^{\mu+1}_q f(z)+q^{\mu}z\partial_q(\zeta^{\mu+1}_q f(z)).$$

We can state that

$$\zeta_q^0 f(z) = z\partial_q f(z),\ \zeta_q' f(z) = f(z) \tag{10}$$

also

$$\lim_{q\to 1^-} \zeta_q^\mu f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k.$$

This means that, by taking $q \to 1^-$, the operator defined in (8) reduces to the famous Noor integral operator given in ([20]). Moreover, for more detailed knowledge on the q-analogue of differential and integral operators, see the work of [20] , [21]. Also for more details on the q-analogue of differential and integral operators, see the work of Aldweby and Darus (see[7])(also see [1, 2, 3, 4, 8], [22]).

Definition 1.2. *Let the functions $h, p : \mathfrak{D} \to \mathbb{C}$ be so constrained that*

$$\min \{\Re (h(z)), \Re (p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

In this study, motivated by the work Altınkaya and Yalçın [6] (see also [5]), [25], by the help of the aforementioned works we introduce a general new subclass $\Im_{\Sigma_\gamma}^{\mu,q}$ of the function class Σ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in our new class . We state that throught this paper, f , g are given by (1) and (2) respectively and ζ_q^μ is $q-$analogue of Noor integral operator.

2 The class $\Im_{\Sigma_\gamma}^{\mu,q}$

Definition 2.1. *For $f \in \Sigma$, $f \in \Im_{\Sigma_\gamma}^{\mu,q}$ if the following conditions are sattisfied:*

$$1 + \frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^\mu f(z)}{z} + \gamma\partial_q(\zeta_q^\mu f(z)) - 1\right] \in h(\mathfrak{D}) \quad ,(z \in \mathfrak{D}), \tag{11}$$

$$1 + \frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^\mu g(w)}{w} + \gamma\partial_q(\zeta_q^\mu g(w)) - 1\right] \in p(\mathfrak{D}) \quad ,(w \in \mathfrak{D}) \tag{12}$$

where $\xi \geq 1, \tau \neq 0$.

Remark 2.2. *There are many choices of h, p and $\mu, q,_\gamma$ which would provide interesting subclasses of class $\Im_{\Sigma_\gamma}^{\mu,q}$. For example*

1. For
$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\lambda},(0<\lambda\leq 1)$$
it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2. It is simply to see that $f\in\Sigma$ is in $\Im^{\mu,q}_{\Sigma_\gamma}(\lambda)$

if the following inequalities hold:

$$\left|\arg\left(1+\tfrac{1}{\tau}\left[(1-\gamma)\tfrac{\zeta_q^\mu f(z)}{z}+\gamma\partial_q(\zeta_q^\mu f(z))-1\right]\right)\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad z\in\mathfrak{D})$$

and

$$\left|\arg\left(1+\tfrac{1}{\tau}\left[(1-\gamma)\tfrac{\zeta_q^\mu g(w)}{w}+\gamma\partial_q(\zeta_q^\mu g(w))-1\right]\right)\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad w\in\mathfrak{D})\ .$$

In the case of $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\lambda}$ where $(0<\lambda\leq 1)$,the class $\Im^{\mu,q}_{\Sigma_\gamma}$ reduces following new also well known subclasses:

i) Upon setting $q\to 1^-$ and it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2. it is simply to see that $f\in\Sigma$ is in
$$\Im^{\mu,1}_{\Sigma_\gamma}(\lambda)=\Im^{\mu}_{\Sigma_\gamma}(\lambda)$$
if the following inequalities hold:
$$\left|\arg\left(1+\tfrac{1}{\tau}\left[(1-\gamma)\tfrac{\zeta^\mu f(z)}{z}+\gamma(\zeta^\mu f(z))'-1\right]\right)\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad z\in\mathfrak{D})$$
and
$$\left|\arg\left(1+\tfrac{1}{\tau}\left[(1-\gamma)\tfrac{\zeta^\mu g(w)}{w}+\gamma(\zeta^\mu g(w))'-1\right]\right)\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad w\in\mathfrak{D})\ .$$

ii) Upon setting $q\to 1^-$, for $\tau=1$,it is simply to see that $f\in\Sigma$ is in
$$\Im^{\mu,1}_{\Sigma_1}(\lambda)=\Im^{\mu}_{\Sigma}(\lambda)$$
if the following inequalities hold:
$$\left|\arg\left[(1-\gamma)\tfrac{\zeta^\mu f(z)}{z}+\gamma(\zeta^\mu f(z))'\right]\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad z\in\mathfrak{D})$$
and
$$\left|\arg\left[(1-\gamma)\tfrac{\zeta^\mu g(z)}{z}+\gamma(\zeta^\mu g(z))'\right]\right|<\frac{\lambda\pi}{2}\quad(0<\lambda\leq 1,\quad z\in\mathfrak{D})\ .$$

iii) Upon setting $q \to 1^-$, for $\tau = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in

$$\Im^{1,1}_{\Sigma_1}(\lambda) = \Im_{\Sigma_1}(\lambda)$$

if the following inequalities hold:

$$\left|\arg\left[(1-\gamma)\tfrac{f(z)}{z} + \gamma(f(z))'\right]\right| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})$$

and

$$\left|\arg\left[(1-\gamma)\tfrac{g(w)}{w} + \gamma(g(w))'\right]\right| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})$$

iv) Upon setting $q \to 1^-$,for $\tau = 1$, $\gamma = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in

$$\Im^{1,1}_{\Sigma_1}(\lambda) = \Im_{\Sigma}(\lambda) = \mathcal{H}^{\lambda}_{\Sigma}$$

The class $\mathcal{H}^{\lambda}_{\Sigma}$ was introduced by Srivastava et al. [27] The function $f \in \mathcal{H}^{\lambda}_{\Sigma}$, if the following inequalities hold:

$$|\arg f'(z)| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})$$

and

$$|\arg g'(z)| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})\,.$$

v) Upon setting $q \to 1^-$,for $\tau = 1$, $\gamma = 1$ and $\mu = 0$, it is simply to see that $f \in \Sigma$ is in

$$\Im^{0,1}_{\Sigma_1}(\lambda) = \Im^{0}_{\Sigma_1}(\lambda)$$

if the following inequalities hold:

$$|\arg(z\partial f(z))'| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})$$

and

$$|\arg(w\partial g(w))'| < \frac{\lambda\pi}{2} \quad (0 < \lambda \le 1, \quad z \in \mathfrak{D})\,.$$

2. For

$$h\,(z) = p\,(z) = \frac{1 + (1-2\beta)\,z}{1-z}, (0 \le \beta < 1)$$

it can be directly verified that the functions $h\,(z)$ and $p\,(z)$ satisfy the hypotheses of Definition 2. It is simply to see that $f \in \Sigma$ is in $\Im^{\mu,q}_{\Sigma_\gamma}(\beta)$, if the following inequalities hold:

$$\Re\left(1 + \tfrac{1}{\tau}\left[(1-\gamma)\tfrac{\zeta^{\mu}_{q} f(z)}{z} + \gamma\partial_q(\zeta^{\mu}_{q} f(z)) - 1\right]\right) > \beta \quad (0 \le \beta < 1,\ 0 < \alpha \le 1,\ z \in \mathfrak{D})$$

and

$$\Re\left(1+\frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^{\mu}g(w)}{w}+\gamma\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)>\beta \quad (0\leq\beta<1,\ 0<\alpha\leq 1,\ w\in\mathfrak{D})\ .$$

In the case of $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ where $(0\leq\beta<1)$,the class $\Im_{\Sigma_\gamma}^{\mu,q}$ reduces following new also well known subclasses:

i) Upon setting $q\to 1^-$ and it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2. it is simply to see that $f\in\Sigma$ is in

$$\Im_{\Sigma_\gamma}^{\mu,1}(\beta)=\Im_{\Sigma_\gamma}^{\mu}(\beta)$$

if the following inequalities hold:

$$\Re\left(1+\frac{1}{\tau}\left[(1-\gamma)\frac{\zeta^{\mu}f(z)}{z}+\gamma(\zeta^{\mu}f(z))'-1\right]\right)>\beta \quad (0\leq\beta<1,\quad z\in\mathfrak{D})$$

and

$$\Re\left(1+\frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^{\mu}g(w)}{w}+\gamma\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)>\beta \quad (0\leq\beta<1,\quad w\in\mathfrak{D})\ .$$

ii) Upon setting $q\to 1^-$, for $\tau=1$,it is simply to see that $f\in\Sigma$ is in

$$\Im_{\Sigma_1}^{\mu,1}(\beta)=\Im_{\Sigma}^{\mu}(\beta)$$

if the following inequalities hold:

$$\Re\left((1-\gamma)\frac{\zeta^{\mu}f(z)}{z}+\gamma(\zeta^{\mu}f(z))'\right)>\beta \quad (0\leq\beta<1,\quad z\in\mathfrak{D})$$

$$\Re\left((1-\gamma)\frac{\zeta^{\mu}g(z)}{z}+\gamma(\zeta^{\mu}g(z))'\right)>\beta \quad (0\leq\beta<1,\quad w\in\mathfrak{D})\,.$$

iii) Upon setting $q\to 1^-$, for $\tau=1$ and $\mu=1$, it is simply to see that $f\in\Sigma$ is in

$$\Im_{\Sigma_1}^{1,1}(\beta)=\Im_{\Sigma_1}(\beta)$$

if the following inequalities hold:

$$\Re\left((1-\gamma)\frac{f(z)}{z}+\gamma(f(z))'\right)>\beta \quad (0\leq\beta<1,\quad z\in\mathfrak{D})$$

and

$$\Re\left((1-\gamma)\frac{g(w)}{w}+\gamma(g(w))'\right)>\beta \quad (0\leq\beta<1,\quad w\in\mathfrak{D})$$

iv) Upon setting $q \to 1^-$,for $\tau = 1$, $\gamma = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in

$$\Im_{\Sigma_1}^{1,1}(\beta) = \Im_\Sigma(\beta) = \mathcal{H}_\Sigma^\beta$$

The class $\mathcal{H}_\Sigma^\beta$ was introduced by Srivastava et al. [27] The function $f \in \mathcal{H}_\Sigma^\beta$, if the following inequalities hold:

$$\Re\left(f'(z)\right) > \beta \quad (0 < \beta \leq 1, \quad z \in \mathfrak{D})$$

and

$$\Re\left(g'(z)\right) > \beta \quad (0 < \beta \leq 1, \quad z \in \mathfrak{D}).$$

v) Upon setting $q \to 1^-$,for $\tau = 1$, $\gamma = 1$ and $\mu = 0$, it is simply to see that $f \in \Sigma$ is in

$$\Im_{\Sigma_1}^{0,1}(\lambda) = \Im_{\Sigma_1}^{0}(\lambda)$$

if the following inequalities hold:

$$|\arg(z\partial f(z))'| > \beta \quad (0 < \lambda \leq 1, \quad z \in \mathfrak{D})$$

and

$$|\arg(w\partial g(w))'| > \beta \quad (0 < \lambda \leq 1, \quad z \in \mathfrak{D}).$$

We need the following lemma to derive our main result.

Theorem 2.3. *Let f given by (1) be in the class $\Im_{\Sigma_\gamma}^{\mu,q}$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{\left(|h'(0)|^2+|p'(0)|^2\right)\tau^2}{(1+\gamma q)^2\phi_1^2}}, \sqrt{\frac{(|h''(0)|+|p''(0)|)\tau}{4(1+\gamma q+\gamma q^2)\phi_2}}\right\} \tag{13}$$

and

$$|a_3| \leq \min\left\{\begin{array}{l} \frac{\left(|h'(0)|^2+|p'(0)|^2\right)}{(1+\gamma q)^2\phi_1^2}\tau^2 + \frac{|h''(0)|+|p''(0)|}{2(1+\gamma q+\gamma q^2)\phi_2}\left|\tau\right|, \\ \\ \frac{|h''(0)|}{(1+\gamma q+\gamma q^2)\phi_2}\left|\tau\right| \end{array}\right. . \tag{14}$$

Proof. Let $f \in \Im_{\Sigma_\gamma}^{\mu,q}$. Firstly, we write the argument inequalities in (11) and (12) and in their equivalent forms as follows:

$$\begin{aligned} 1+\frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^\mu f(z)}{z}+\gamma\partial_q(\zeta_q^\mu f(z))-1\right] &= h(z) \\ 1+\frac{1}{\tau}\left[(1-\gamma)\frac{\zeta_q^\mu g(w)}{w}+\gamma\partial_q(\zeta_q^\mu g(w))-1\right] &= p(w) \end{aligned} \tag{15}$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definiton 2. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$\begin{aligned} h(z) &= 1 + h_1 z + h_2 z^2 + \cdots\ , \\ p(w) &= 1 + p_1 w + p_2 w^2 + \cdots\ , \end{aligned} \tag{16}$$

respectively. Thus, upon comparing the corresponding coefficients in (15), we get

$$\frac{1}{\tau}(1+\gamma q)\,\phi_1 a_2 = h_1 \quad ... \tag{17}$$

$$\frac{1}{\tau}(1+\gamma q+\gamma q^2)\phi_2 a_3 = h_2\ ... \tag{18}$$

and

$$-\frac{1}{\tau}(1+\gamma q)\,\phi_1 a_2 = p_1.. \tag{19}$$

$$\frac{1}{\tau}\left[(1+\gamma q+\gamma q^2)\phi_2\left(2a_2^2 - a_3\right)\right] = p_2\ .... \tag{20}$$

From (17) and (19), we have

$$h_1 = -p_1 \quad ... \tag{21}$$

and

$$2\left[\frac{1}{\tau}(1+\gamma q)\,\phi_1\right]^2 a_2^2 = h_1^2 + p_1^2. \quad ... \tag{22}$$

Also from (18) and (20), we get

$$\frac{2}{\tau}\left[(1+\gamma q+\gamma q^2)\phi_2\right] a_2^2\ = h_2 + p_2.... \tag{23}$$

which gives us the desired estimate on $|a_2|$ as asserted in (13).

Next, in order to find the bound on $|a_3|$, by subtracting (20) from (18), we obtain

$$a_3 - a_2^2 = \frac{h_2 - p_2}{2(1+\gamma q+\gamma q^2)\phi_2}\tau \tag{24}$$

Therefore, in view of (22) and (23), we have

$$a_3 = \frac{h_2 - p_2}{2(1+\gamma q+\gamma q^2)\phi_2}\tau + \frac{(h_1^2 + p_1^2)}{2(1+\gamma q)^2\phi_1^2}\tau^2$$

which completes the proof of Theorem. □

3 Corollaries and Consequences

Corollary 3.1. *If we let*

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\lambda} = 1 + 2\lambda z + 2\lambda^2 z^2 + ... \qquad (0 < \lambda \leq 1),$$

then inequalities (13) and (14) become

$$|a_2| \leq \min\left\{\tfrac{2\lambda|\tau|}{(1+\gamma q)\phi_1}, \sqrt{\tfrac{2\lambda\tau}{(1+\gamma q+\gamma q^2)\phi_2}}\right\} = \sqrt{\tfrac{2\lambda\tau}{(1+\gamma q+\gamma q^2)\phi_2}} \tag{25}$$

and

$$|a_3| \leq \min\left\{\frac{4\lambda^2\tau^2}{(1+\gamma q)^2\phi_1^2} + \tfrac{2\lambda|\tau|}{(1+\gamma q+\gamma q^2)\phi_2}, \tfrac{2\lambda|\tau|}{(1+\gamma q+\gamma q^2)\phi_2}\right\}. \tag{26}$$

Corollary 3.2. *If we let*

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0 \leq \beta < 1),$$

then inequalities (13) and (14) become

$$|a_2| \leq \min\left\{\frac{2(1-\beta)|\tau|}{(1+\gamma q)\phi_1}, \sqrt{\frac{2(1-\beta)\tau}{(1+\gamma q+\gamma q^2)\phi_2}}\right\}.$$

and

$$|a_3| \leq \min\left\{\frac{4(1-\beta)^2\tau^2}{(1+\gamma q)^2\phi_1^2} + \frac{2(1-\beta)|\tau|}{(1+\gamma q+\gamma q^2)\phi_2}, \frac{2(1-\beta)|\tau|}{(1+\gamma q+\gamma q^2)\phi_2}\right\}.$$

An urgent and important corollary of the Theorem for $q \to 1^-$ is asserted by following Corollary.

Corollary 3.3. *If $f \in \Im_{\Sigma_\gamma}^{\mu,q}$, then*

$$|a_2| \leq \min\left\{\sqrt{\tfrac{\left(|h'(0)|^2+|p'(0)|^2\right)\tau^2}{(1+\gamma)^2\phi_1^2}}, \sqrt{\tfrac{(|h''(0)|+|p''(0)|)\tau}{4(1+2\gamma)\phi_2}}\right\} \tag{27}$$

and

$$|a_3| \leq \min\left\{\begin{array}{l}\tfrac{\left(|h'(0)|^2+|p'(0)|^2\right)}{(1+\gamma)^2\phi_1^2}\tau^2 + \tfrac{|h''(0)|+|p''(0)|}{2(1+2\gamma)\phi_2}|\tau|, \\ \\ \tfrac{|h''(0)|}{(1+2\gamma)\phi_2}|\tau|\end{array}\right. . \tag{28}$$

Corollary 3.4. *If we let*

$$h\left(z\right)=p\left(z\right)=\left(\frac{1+z}{1-z}\right)^{\lambda}=1+2\lambda z+2\lambda^{2}z^{2}+...\qquad\left(0<\lambda\leq 1\right),$$

then inequalities (27) and (28) become

$$|a_2|\leq \min\left\{\tfrac{2\lambda|\tau|}{(1+\gamma)\phi_1},\sqrt{\tfrac{2\lambda\tau}{(1+2\gamma)\phi_2}}\right\}=\sqrt{\tfrac{2\lambda\tau}{(1+2\gamma)\phi_2}}\tag{29}$$

and

$$|a_3|\leq \min\left\{\frac{4\lambda^2\tau^2}{(1+\gamma)^2\phi_1^2}+\tfrac{2\lambda|\tau|}{(1+2\gamma)\phi_2},\tfrac{2\lambda|\tau|}{(1+2\gamma)\phi_2}\right\}.\tag{30}$$

By choosing $\gamma=1$ in inequalities (29) and (30) we get

$$|a_2|\leq \min\left\{\tfrac{\lambda|\tau|}{\phi_1},\sqrt{\tfrac{2\lambda\tau}{3\phi_2}}\right\}=\sqrt{\tfrac{2\lambda\tau}{3\phi_2}}$$

and

$$|a_3|\leq \min\left\{\frac{\lambda^2\tau^2}{\phi_1^2}+\tfrac{2\lambda|\tau|}{3\phi_2},\tfrac{2\lambda|\tau|}{3\phi_2}\right\}.$$

Corollary 3.5. *If we let*

$$h\left(z\right)=p\left(z\right)=\frac{1+\left(1-2\beta\right)z}{1-z}=1+2\left(1-\beta\right)z+2\left(1-\beta\right)z^{2}+\cdots\quad\left(0\leq\beta<1\right),$$

then inequalities (27) and (28) become

$$|a_2|\leq \min\left\{\frac{2\left(1-\beta\right)|\tau|}{(1+\gamma)\phi_1},\sqrt{\frac{2(1-\beta)\tau}{(1+2\gamma)\phi_2}}\right\}.\tag{31}$$

and

$$|a_3|\leq \min\left\{\frac{4\left(1-\beta\right)^2\tau^2}{(1+\gamma)^2\phi_1^2}+\frac{2\left(1-\beta\right)|\tau|}{(1+2\gamma)\phi_2},\frac{2\left(1-\beta\right)|\tau|}{(1+2\gamma)\phi_2}\right\}.\tag{32}$$

By choosing $\gamma=1$ in inequalities (31) and (32) we get

$$|a_2|\leq \min\left\{\frac{\left(1-\beta\right)|\tau|}{\phi_1},\sqrt{\frac{2(1-\beta)\tau}{3\phi_2}}\right\}.\tag{33}$$

and

$$|a_3|\leq \min\left\{\frac{\left(1-\beta\right)^2\tau^2}{\phi_1^2}+\frac{2\left(1-\beta\right)|\tau|}{3\phi_2},\frac{2\left(1-\beta\right)|\tau|}{3\phi_2}\right\}.\tag{34}$$

4 Coclusion

In this paper, we defined a new subclasses of bi-univalent functions in the open unit disk D, by usinq Q- analoquo of Noor integral operator. . For the functions belonging to this subclass, the estimates of second and third Taylor–Maclaurin coefficients are obtained. Also, some interesting corollaries and remarks are discussed and relevant connections are made..

5 Open Problem

For the subclass $\Im_{\Sigma_\gamma}^{\mu,q}$, it is interesting to obtain the upper bounds of the Fekete-Szegö inequality, and the bounds of the Hankel determinants. Also, to examine the estimates of second and third Taylor-Maclaurin coefficients $m-$fold symmetric bi-univalent functions.

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