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Dual ruled surface constructed by the pole curve of the involute curve

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Abstract

In this paper, the dual geometric invariants of the closed ruled surface $\left[\overline{C}\right]$ corresponding to the pole curve $\overline{C}$ of the dual involute curve are investigated and some new interesting results about the developability of this surface are also given. In addition, spherical areas limited by the dual ruled srfaces $[R_1], [R_2], [R_3]$ corresponding to Frenet vectors $\{R_1, R_2, R_3\}$ of the dual involute curve and $\left[\overline{C}\right]$ are calculated. Finally such surfaces are illustrated with one example.

Keywords: *Closed Ruled Surface, Dual Involute-Evolute, Integral Invariant, Pole Curve, Spherical Area.*

2010 Mathematics Subject Classification: 51M30, 53A17, 53A25.

1 Introduction

The idea of dual number was initially submitted by William Kingdon Clifford (1845-1879), [5]. After him German mathematician Eduard Study used these numbers in his survey on the theory of surface and mechanic [15]. According to the basic principle of Study, the dual points of the dual unit sphere $S^2_{\mathbb{D}}$ correspond exactly to the directional lines in $\mathbb{R}^3$. Hence, a differentiable curve on the sphere $S^2_{\mathbb{D}}$ corresponds to a ruled surface in the line space (see [7]). This act is called as E. Study Mapping. Inspired by the Study transformatin in dual space, many studies have been done on the geometric invariants of the closed ruled surface. In this context, the geometric elements of the closed ruled surface related the one parameter dual unit spherical curves in space of lines $\mathbb{R}^3$ were studied by Hacısalihoglu and Gursoy [8, 9, 10, 11], respectively.

There are many curves derived from a curve. One of the most important of these is the involute curve. Involute is a curve that intersects all the tangents of a given curve at right angle. For the first time in the literature, the angular relations between the Frenet vektors of the involute-evolute curve couple have been expressed in both Euclidean space and Lorentz space, [1, 2, 3, 6]. And also the relationships between dual Frenet frames and Darboux vectors of the involute-evolute curve couple were found in the dual space [17]. In consideration of the literature, dual vector analysis, space curves and dual ruled surface have been studied by many authors [12, 13, 14, 16, 18, 19, 20]. As well as the results of integral invariants of closed ruled surface corresponding to Frenet vectors $\{R_1,\ R_2,\ R_3\}$ of the involute curve in the dual space are also given by [4].

Unlike the available literature in this study, we will introduce some geometric invariants of the closed ruled surface corresponding to the pole curve of the involute curve in dual space. Also, we give the dual spherical areas bounded by the dual closed ruled surfaces $[R_1], [R_2], [R_3]$ and $\left[\overline{C}\right]$. Finally, we illustrate the dual ruled surfaces generated by the Frenet vectors and the pole curve of the dual involute curve with one example.

2 Preliminaries

The set of $\mathbb{R} \times \mathbb{R}$, which consists of ordered pairs (x, x^*), each x and $x^* \epsilon \mathbb{R}$, is called the dual number system $\mathbb{D}$. Thus, a dual number can be expressed as $X = x + \varepsilon x^*$, where $\varepsilon = (0, 1)$ is the dual unit element that satisfies $\varepsilon^2 = 0$. $\mathbb{D}$ is a commutative ring with the unit element 1. $\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D}$ is a module on $\mathbb{D}$. Each element of the $\mathbb{D} - Module$ is called a dual vector and is expressed as $\overrightarrow{X} = (X_1, X_2, X_3) = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*) = \overrightarrow{x} + \varepsilon \overrightarrow{x^*}$.

For $\overrightarrow{A}$ and $\overrightarrow{B} \in \mathbb{D}^3$, the consepts inner product, outer product, norm and dual angle are defined as, respectively.

$$\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle = \langle \overrightarrow{x}, \overrightarrow{y} \rangle + \varepsilon \left(\left\langle \overrightarrow{x}, \overrightarrow{y^*} \right\rangle + \left\langle \overrightarrow{x^*}, \overrightarrow{y} \right\rangle \right), \tag{1}$$

$$\overrightarrow{X} \times \overrightarrow{Y} = \overrightarrow{x} \times \overrightarrow{y} + \varepsilon\left(x \times y^{*} + x^{*} \times y\right), \tag{2}$$

$$\left\|\overrightarrow{X}\right\| = \sqrt{\left\langle \overrightarrow{X}, \overrightarrow{X} \right\rangle} = \left\|\overrightarrow{x}\right\| + \varepsilon \frac{\left\langle \overrightarrow{x}, \overrightarrow{x^{*}} \right\rangle}{\left\|\overrightarrow{x}\right\|} \tag{3}$$

and the dual angle is related to the inner product as follow

$$\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle = \cos\Phi = \cos\phi - \varepsilon\phi^{*}\cos\phi, \tag{4}$$

where $\Phi = \phi + \varepsilon\phi^{*}$, $0 \leq \phi \leq \pi$, and $\phi^{*} \in \mathbb{R}$ is a dual number. The geometric location of the points one unit away from a fixed point $(0,0,0) \in \mathbb{D}$ is called the unit dual sphere and is defined as $S^{2}_{\mathbb{D}} = \left\{ \overrightarrow{X} = \overrightarrow{x} + \varepsilon\overrightarrow{x^{*}} : \left\|\overrightarrow{x}\right\| = (1,0) \right\}$, [11].

Let's immediately note that all the curves examined in this article are closed. $X : I \subset \mathbb{R} \longrightarrow \mathbb{D}^{3}$, $s \rightarrow X(s) = x(s) + \varepsilon x^{*}(s)$ is differentiable in dual space $\mathbb{D}^{3}$. The dual arc-length of the $X(s)$ from s_1 to s is defined by

$$\widetilde{s} = \int_{s_1}^{s} \left\| x^{'}(s) \right\| ds + \int_{s_1}^{s} \left\langle t, x^{*^{'}}(s) \right\rangle = s + \varepsilon s^{*}$$

where t is unit tangent vector of the indicatrix $x(s)$ [11].

Let $\{T, N, B\}$ be the moving dual Frenet frame of the curve $X' = T$, with

$$T = X', \ N = \frac{X''}{\|X''\|}, \ B = T \times N.$$

The Frenet formulas for dual curves can be given as follows

$$\begin{bmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{5}$$

where $\kappa(s) = k_1 + \varepsilon k_1^{*}$ and $\tau(s) = k_2 + \varepsilon k_2^{*}$ are the dual curvature and torsion of $X(s)$, respectively, [13].

Let Ψ is instantaneos dual Pfaff vector of X. This vector is determined with the equation

$$\Psi = \tau T + \kappa B. \tag{6}$$

The pole curve C of the unit dual curve X

$$C(s) = \frac{\tau}{\|\Psi\|} T + \frac{\kappa}{\|\Psi\|} B \tag{7}$$

If the κ and τ values are written instead, we obtain

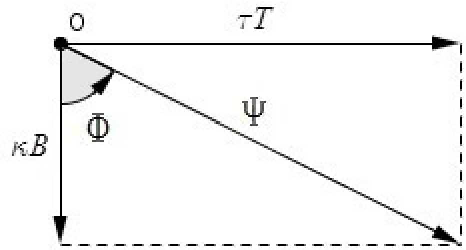

Figure 1: Dual Pfaff vector

$$C(s) = \sin\Phi T + \cos\Phi B. \tag{8}$$

The Steiner vector is defined as

$$D = d + \varepsilon d^* = \Psi, \tag{9}$$

the angle of the pitch of ($closedruledsurface$) $[X]$ is given by

$$\Lambda_X = -\langle D, X\rangle = \lambda_X - \varepsilon L_X, \tag{10}$$

here L_X and λ_X are the intergal invariants of $[X]$, [8].

Theorem 2.1 *Let $\{X\}$ be the dual orbit on K' of an arbitrary fixed dual point X on K. The dual spherical area bounded by the dual closed spherical curve X is calculated as*

$$F_X = 2\pi(1-n) - \langle D, X\rangle. \tag{11}$$

Here n is the rotation number of the rotation of the pole $\{C\}$ at the point X and X denotes the dual position vector of an arbitrary point of the dual closed curve $\{X\}$ on K' [14].

If $\hat{\beta}$ is involute of the dual curve $\hat{\alpha}$ we can write,

$$\langle T, R_1\rangle = 0, \tag{12}$$

where T, R_1 are tangents of the curves $\hat{\alpha}$ and $\hat{\beta}$, respectively. According to this definition, If the curve $\hat{\beta}$ is involute of $\hat{\alpha}$, we can write $\hat{\beta}(s) = \hat{\alpha}(s) + [(c-s)+\varepsilon d]\, T(s)$, $\quad c,\ d \in \mathbb{R}$ [17]. The relations between the Frenet frames of the curve couple $\left(\hat{\alpha}, \hat{\beta}\right)$ as follow [17].

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\overline{\Phi} & 0 & \sin\overline{\Phi} \\ \sin\overline{\Phi} & 0 & \cos\overline{\Phi} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{13}$$

The real and dual parts of R_1, R_2, R_3 are

$$\begin{cases} r_1 = n \\ r_2 = -\cos\overline{\phi}t + \sin\overline{\phi}b \\ r_3 = \sin\overline{\phi}t + \cos\overline{\phi}b \\ r_1^* = n^* \\ r_2^* = -\cos\overline{\phi}t^* + \sin\overline{\phi}b^* + \overline{\phi}^*\left(\sin\overline{\phi}t + \cos\overline{\phi}b\right) \\ r_3^* = \sin\overline{\phi}t^* + \cos\overline{\phi}b^* + \overline{\phi}^*\left(\cos\overline{\phi}t - \sin\overline{\phi}b\right) \end{cases}, \tag{14}$$

where $\overline{\Phi} = \overline{\phi} + \varepsilon\overline{\phi}^*$ is the dual angle between the $\overline{\Psi}$ and R_2. Derivative formulas for the Equation (13) can be given as follows.

$$\begin{cases} R_1' = PR_2 \\ R_2' = -PR_1 + QR_3 \\ R_3' = -QR_2 \end{cases}, \tag{15}$$

here $P = (p, p^*)$ and $Q = (q, q^*)$ are the dual curvatures of the $\hat{\beta}$. The real and dual elements of Equation (15) are

$$\begin{cases} r_1' = pr_2 \\ r_2' = -pr_1 + qr_3 \\ r_3' = -qr_2 \\ r_1^{*\prime} = p^*r_2 + pr_2^* \\ r_2^{*\prime} = -p^*r_1 + -pr_1^* + q^*r_3 + qr_3^* \\ r_3^* = -q^*r_2 - qr_2^* \end{cases}. \tag{16}$$

Since $P = \frac{\left\|\hat{\beta}' \wedge \hat{\beta}''\right\|}{\left\|\hat{\beta}'\right\|^3}$, we have that $P = \frac{\sqrt{\kappa^2+\tau^2}}{\mu\kappa}$. Using the formula $Q = \frac{\det\left(\hat{\beta}', \hat{\beta}'', \hat{\beta}'''\right)}{\left\|\hat{\beta}' \wedge \hat{\beta}''\right\|^2}$, we find $Q = \frac{\overline{\Phi}'}{\mu\kappa}$ [10]. The real and dual parts of P and Q are

$$\begin{cases} p = \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1} \\ p^* = \frac{k_2\left(k_1k_2^* - k_1^*k_2\right)}{\mu k_1^2\sqrt{k_1^2+k_2^2}} \\ q = \frac{\overline{\phi}'}{\mu k_1} \\ q^* = \frac{k_1^{*\prime}\overline{\phi}^{*\prime} - k_1^*\overline{\phi}'}{\mu k_1^2} \end{cases}. \tag{17}$$

3 Main results

In this section, we first examine some goemetric invariants of the closed ruled surface $\left[\overline{C}\right]$ contructed by the pole curve $\overline{C}$ of the dual involute curve $\hat{\beta}$. Then we give some new interesting results about the developability of this surface. In addition, spherical areas limited by the dual closed ruled surface $[R_1], [R_2], [R_3]$

and $\left[\overline{C}\right]$ are calculated and these closed ruled surfaces are illustrated with one example.

The dual rotation vector $\bar{\Psi}$and the pole curve $\overline{C}$ of the involute curve $\hat{\beta}$ is determined by, respectively

$$\overline{\Psi} = QR_1 + PR_3, \tag{18}$$

$$\overline{C} = \frac{Q}{\left\|\overline{\Psi}\right\|}R_1 + \frac{P}{\left\|\overline{\Psi}\right\|}R_3. \tag{19}$$

Derivation of $\overline{C}$ according to s, we have

$$\frac{d\overline{C}}{ds} = \frac{\left(Q^{'}R_1 + QR_1^{'} + PR_3^{'} + P^{'}R_3\right)\left\|\overline{\Psi}\right\| - \left\|\overline{\Psi}\right\|^{'}\left(QR_1 + PR_3\right)}{\left\|\overline{\Psi}\right\|^2}. \tag{20}$$

The real and dual parts of Equation (20) are

$$\frac{d\overline{c}}{ds} = \frac{r_1\left(p^2q' - pp'q\right) + r_3\left(q^2p' - qq'p\right)}{\left\|\overline{\Psi}\right\|^3}, \tag{21}$$

$$\begin{aligned} \frac{d\overline{c}^*}{ds} &= \frac{r_1^*\left(p^2q' - pp'q\right) + r_3^*\left(q^2p' - qq'p\right)}{\left\|\overline{\Psi}\right\|^3} \\ &\quad + \frac{r_1\left(p^2q^{*\prime} + 2pp^*q' - pp'q^* - qp'p^* - qpp^{*\prime}\right)}{\left\|\overline{\Psi}\right\|^3} \\ &\quad + \frac{r_3\left(q^2p^{*\prime} + 2qq^*p' - qq'p^* - pq'q^* - pqq^{*\prime}\right)}{\left\|\overline{\Psi}\right\|^3}. \end{aligned} \tag{22}$$

The dual Steiner vector of the curve $\hat{\beta}$ is

$$\overline{D} = R_1Qds + R_3Pds. \tag{23}$$

The real and dual elements of Equation (23) are

$$\begin{cases} \overline{d} = r_1qds + r_3pds \\ \overline{d}^* = r_1q^*ds + r_1^*qds + r_3p^*ds + r_3^*pds \end{cases}. \tag{24}$$

The dual angle of the pitch of the closed ruled surface $\left[\overline{C}\right]$ is given by

$$\Lambda_{\overline{C}} = -\left\langle \overline{D}, \overline{C} \right\rangle. \tag{25}$$

Substituting (19), (23) into (25), we have

$$\Lambda_{\overline{C}} = -\left\langle R_1 Q ds + R_3 P ds, \sin\bar{\Phi} R_1 + \cos\bar{\Phi} R_3 \right\rangle, \tag{26}$$

$$\begin{aligned} \Lambda_{\overline{C}} \;\; = \;\; & -\sin\overline{\phi} q ds - \cos\overline{\phi} p ds \\ & -\varepsilon(\overline{\phi}^{*}\cos\overline{\phi} q ds + \sin\overline{\phi} q^{*} ds 3.10 \\ & -\overline{\phi}^{*}\sin\overline{\phi} p ds + \cos\overline{\phi} p^{*} ds). \end{aligned} \tag{27}$$

Substituting the Equation (17) into Equation (27), we reach

$$\begin{aligned} \Lambda_{\overline{C}} \;\; = \;\; & -\sin\phi \frac{\bar{\phi}'}{\mu k_1} ds - \cos\phi \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1} ds \\ & -\varepsilon(\bar{\phi}^{*}\cos\bar{\phi} \frac{\overline{\phi}'}{\mu k_1} ds + \sin\bar{\phi} \frac{k_1^{*\prime}\overline{\phi} - k_1^{*}\overline{\phi}'}{\mu k_1^2} ds 3.11 \\ & -\overline{\phi}^{*}\sin\overline{\phi} \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1} ds + \cos\overline{\phi} \frac{k_2\left(k_1 k_2^{*} - k_1^{*} k_2\right)}{\mu k_1^2 \sqrt{k_1^2+k_2^2}} ds). \end{aligned} \tag{28}$$

Corollary 3.1 *The angle of pitch and the pitch of the closed ruled surface* $\left[\overline{C}\right]$ *are*

$$\lambda_{\overline{C}} = -\sin\phi \frac{\overline{\phi}'}{\mu k_1} ds - \cos\phi \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1} ds$$

and

$$\begin{aligned} L_{\overline{C}} \;\; = \;\; & \overline{\phi}^{*}\cos\overline{\phi} \frac{\overline{\phi}'}{\mu k_1} ds + \sin\overline{\phi} \frac{k_1 \overline{\phi}^{*\prime} - k_1^{*}\overline{\phi}'}{\mu k_1^2} ds \\ & -\overline{\phi}^{*}\sin\overline{\phi} \frac{\sqrt{k_1^2+k_2^2}}{\mu k_1} ds + \cos\overline{\phi} \frac{k_2\left(k_1 k_2^{*} - k_1^{*} k_2\right)}{\mu k_1^2 \sqrt{k_1^2+k_2^2}} ds \end{aligned}$$

respectively.

The drall of the closed ruled surface $\left[\overline{C}\right]$ is

$$\Delta_{\overline{C}} = \frac{\langle d\overline{c}, d\overline{c}^{*}\rangle}{\langle d\overline{c}, d\overline{c}\rangle}. \tag{29}$$

Using the values of $d\overline{c}$ and $d\overline{c}^{*}$ in Equation (29), gives

$$\Delta_{\overline{C}} = pq^{*'} - qp^{*'} + p^{*}q' - q^{*}p' + \frac{\left(pq' - qp'\right)\left(q^{*}q + p^{*}p\right)}{p^2+q^2}. \tag{30}$$

Theorem 3.2 *The closed ruled surface* $\left[\overline{C}\right]$ *generated by* $\overline{C}$ *is developable if and only if* $\frac{p}{q}=\frac{p^{*\prime}}{q^{*\prime}}=-\frac{q^{*}}{p^{*}}=-\frac{q^{\prime}}{p^{\prime}}$ *or* $\frac{p}{q}=\frac{p^{*\prime}}{q^{*\prime}}=\frac{p^{*}}{q^{*}}=\frac{p^{\prime}}{q^{\prime}}$.

Corollary 3.3 *The closed ruled surface* $\left[\overline{C}\right]$ *generated by* $\overline{C}$ *is developable, using the values of* p,q,p^{*} *and* q^{*} *in (2.17) into the last equation, we get*

$$\begin{aligned}
\Delta_{\overline{C}} \;=\;& \frac{\sqrt{k_1^2+k_2^2}\left[\mu\left(k_1\left(k_1^{*\prime\prime}\overline{\phi}-\overline{\phi}^{\prime\prime}k_1^{*}\right)+\overline{\phi}^{\prime}\left(k_1^{\prime}k_1^{*}-k_1k_1^{*\prime}\right)+k_1^{\prime}\left(k_1^{*}\overline{\phi}^{\prime}-k_1^{*\prime}\overline{\phi}^{\prime}\right)\right)\right]}{\mu^3k_1^4}\\
&+\frac{\sqrt{k_1^2+k_2^2}\mu^{\prime}k_1^2\left(k_1\overline{\phi}^{\prime}-\overline{\phi}^{*\prime}\right)}{\mu^3k_1^4}\\
&+\frac{\overline{\phi}^{\prime}\left[\mu k_1^2\left(k_1^2+k_2^2\right)\left(k_2\left(k_1k_2^{*^{\prime}}-k_1^{*^{\prime}}k_2\right)+k_2\left(k_1^{\prime}k_2^{*}-k_1^{*}k_2^{\prime}\right)+k_2^{\prime}\left(k_1k_2^{*}-k_1^{*}k_2\right)\right)\right]}{\mu^3k_1^5\sqrt[3]{k_1^2+k_2^2}}\\
&-\frac{\overline{\phi}^{\prime}\left[\left(\mu^{\prime}k_1^2\left(k_1^2+k_2^2\right)+2\mu k_1k_1^{\prime}\left(k_1^2+k_2^2\right)+\mu k_1^2\left(k_1k_1^{\prime}+k_2k_2^{\prime}\right)\right)k_2\left(k_1k_2^{*}-k_1^{*}k_2\right)\right]}{\mu^3k_1^5\sqrt[3]{k_1^2+k_2^2}}\\
&+\frac{k_2\left(k_1k_2^{*}-k_1^{*}k_2\right)\left[\mu\left(\overline{\phi}^{\prime\prime}k_1-k_1^{\prime}\overline{\phi}^{\prime}\right)-\mu^{\prime}k_1\overline{\phi}^{\prime}\right]}{\mu^3k_1^4\sqrt{k_1^2+k_2^2}}\\
&-\frac{\left(k_1\overline{\phi}^{*\prime}-k_1^{*}\overline{\phi}^{\prime}\right)\left[\mu k_2\left(k_1k_2^{\prime}-k_1^{\prime}k_2\right)-\mu^{\prime}k_1\left(k_1^2+k_2^2\right)\right]}{\mu^3k_1^4\sqrt{k_1^2+k_2^2}}\\
&+\frac{\left[\left(k_1\overline{\phi}^{*\prime}-k_1^{*}\overline{\phi}^{\prime}\right)\sqrt{k_1^2+k_2^2}+k_2\left(k_1k_2^{*}-k_1^{*}k_2\right)\right]\left[\left(k_1^2+k_2^2\right)\left(\overline{\phi}^{\prime\prime}k_1-k_1^{\prime}\overline{\phi}^{\prime}\right)\right]}{\mu^2k_1^4\left(k_1^2+k_2^2+\bar{\phi}^{\prime 2}\right)\sqrt{k_1^2+k_2^2}}\\
&+\frac{\left[\left(k_1\overline{\phi}^{*\prime}-k_1^{*}\overline{\phi}^{\prime}\right)\sqrt{k_1^2+k_2^2}+k_2\left(k_1k_2^{*}-k_1^{*}k_2\right)\right]\overline{\phi}^{\prime}k_2\left(k_1^{\prime}k_2-k_1k_2^{\prime}\right)}{\mu^2k_1^4\left(k_1^2+k_2^2+\overline{\phi}^{\prime 2}\right)\sqrt{k_1^2+k_2^2}}.
\end{aligned} \tag{31}$$

Theorem 3.4 *The closed ruled surface* $\left[\overline{C}\right]$ *generated by* $\overline{C}$ *is developable if and only if* $k_1=\frac{\overline{\phi}^{*\prime}}{\overline{\phi}^{\prime}}$ *and* $\frac{k_1}{k_2}=\frac{k_1^{*}}{k_2^{*}}=\frac{k_1^{*^{\prime}}}{k_2^{*^{\prime}}}=\frac{k_1^{\prime}}{k_2^{\prime}},\frac{k_1}{k_1^{*}}=\frac{\overline{\phi}^{\prime}}{\overline{\phi}^{*\prime}}=\frac{\overline{\phi}^{\prime\prime}}{\overline{\phi}^{*\prime\prime}}$.

Using Eq. (8), we have

$$\overline{C}(s)=\sin\overline{\Phi}R_1+\cos\overline{\Phi}R_3. \tag{32}$$

Separate (32) into real and dual components, we can write following two equations

$$\begin{cases} \overline{c}(s) = r_1 \sin \overline{\phi} + r_3 \cos \overline{\phi} \\ \overline{c}^*(s) = r_1 \overline{\phi}^* \cos \overline{\phi} + r_1^* \sin \overline{\phi} + r_3 \cos \overline{\phi} - r_3^* \overline{\phi}^* \sin \overline{\phi} \end{cases} \tag{33}$$

Let's take a derivative of $\overline{c}(s)$ and $\overline{c}^*(s)$ according to s,

$$\frac{d\overline{c}}{ds} = r_1' \overline{\phi}' \cos \bar{\phi} - r_3 \overline{\phi}' \sin \overline{\phi} \tag{34}$$

$$\begin{aligned} \frac{d\overline{c}^*}{ds} &= r_1 \left(\overline{\phi}^{*\prime} \cos \overline{\phi} - \overline{\phi}^* \overline{\phi}' \sin \overline{\phi} \right) + r_1^* \overline{\phi}' \cos \overline{\phi} \\ &+ r_2 \left(q \overline{\phi}^* \cos \overline{\phi} - p \cos \overline{\phi} + q^* \sin \overline{\phi} + p^* \overline{\phi}^* \sin \overline{\phi} \right) \\ &+ r_2^* \left(q \sin \overline{\phi} + p \overline{\phi}^* \sin \overline{\phi} \right) + r_3 \left(- \overline{\phi}' \sin \overline{\phi} \right) 3.18 \\ &+ r_3^* \left(- \overline{\phi}^{*\prime} \sin \overline{\phi} - \overline{\phi}^* \overline{\phi}' \cos \overline{\phi} \right) \end{aligned} \tag{35}$$

From the above equality (34), (35) and (29) ,we obtain

$$\Delta_{\overline{C}} = \frac{-q \sin \overline{\phi} - p \overline{\phi}^* \sin \overline{\phi} + \overline{\phi}^{*\prime} \cos^2 \overline{\phi} - \overline{\phi}^* \overline{\phi}' \sin \overline{\phi} \cos \overline{\phi} + \overline{\phi}' \sin^2 \overline{\phi}}{\overline{\phi}'}. \tag{36}$$

Theorem 3.5 *The closed ruled surface* $\left[\overline{C}\right]$ *generated by* $\overline{C}$ *is developable if and only if* $\overline{\phi}' \neq 0$, $\overline{\phi}^* = 0$ *and* $\overline{\phi} = \arcsin \frac{1}{\mu k_1}$.

The dual angle of the pitch of the closed ruled surfaces $[R_1]$,$[R_2]$,$[R_3]$ is as follow, respectively.

$$\Lambda_{R_1} = -Q ds, \tag{37}$$

$$\Lambda_{R_2} = 0. \tag{38}$$

$$\Lambda_{R_3} = -P ds, [5]. \tag{39}$$

If these last equations, (25) and Theorem 2.1 are used together, dual spherical areas bounded by the dual closed ruled surfaces $[R_1]$,$[R_2]$,$[R_3]$ and $\left[\overline{C}\right]$ can be calculated as follows

$$F_{R_1} = 2\pi \left(1 - n\right) - Q ds, \tag{40}$$

$$F_{R_2} = 2\pi \left(1 - n\right), \tag{41}$$

$$F_{R_3} = 2\pi(1-n) - Pds, \tag{42}$$

$$\begin{aligned} F_{\overline{C}} &= 2\pi(1-n) - \sin\overline{\phi}qds - \cos\overline{\phi}pds \\ &-\varepsilon(\overline{\phi}^* \cos\overline{\phi}qds + \sin\overline{\phi}q^*ds \\ &-\overline{\phi}^* \sin\overline{\phi}pds + \cos\overline{\phi}p^*ds). \end{aligned} \tag{43}$$

Considering the dual curvatures P, Q of the dual involute curve and dual angle $\overline{\Phi}$, the following important conclusion can be given regarding dual spherical areas bounded by the dual closed ruled surfaces $[R_1]$,$[R_2]$,$[R_3]$ and $[\overline{C}]$

1. If $Q = 0$ then $F_{R_1} = F_{R_2}$.
2. If $P = 0$ then $F_{R_2} = F_{R_3}$.
3. If $\bar{\Phi} = 0$ then $F_{R_3} = F_{\overline{C}}$.
4. If $Q = 0$ and $\overline{\phi} = 0$ then $F_{R_1} = F_{R_2}$ and $F_{R_3} = F_{\overline{C}}$.
5. If $P = 0$ and $\overline{\phi} = 0$ then $F_{R_2} = F_{R_3}$.
6. If $P = 0$ and $\overline{\Phi} = 0$ then $F_{R_2} = F_{R_3} = F_{\overline{C}}$.
7. If $P = 0$ and $\overline{\phi} = \frac{\pi}{2}$ then $F_{R_1} = F_{\overline{C}}$ and $F_{R_2} = F_{R_3}$.

4 Example

Example 1 : Let $\widehat{\alpha}(s) = \left(\frac{1}{2}\cos s, -\sin s, \frac{\sqrt{3}}{2}\cos s\right) + \varepsilon\left(\frac{1}{2}\sin s, \cos s, \frac{\sqrt{3}}{2}\sin s\right)$ be a dual space curve and its tangent

$$T(s) = \left(-\frac{1}{2}\sin s, -\cos s, -\frac{\sqrt{3}}{2}\sin s\right) + \varepsilon\left(\frac{1}{2}\cos s, -\sin s, \frac{\sqrt{3}}{2}\cos s\right).$$

The dual arc-length of the dual space curve $\widehat{\alpha}(s)$ from 0 to s is

$$\tilde{s} = \int_0^s \left\|\alpha'(s)\right\| ds + \int_0^s \left\langle t, \alpha^{*'}(s)\right\rangle = s, where \left\|\widehat{\alpha}'(s)\right\| = 1.$$

From Theorem 3.1

$$\begin{aligned} \widehat{\beta}(s) &= \left(\frac{1}{2}\cos s + \frac{1}{2}(c-s)\sin s, -\sin s - (c-s)\cos s, \frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}(c-s)\cos s\right) \\ &+\varepsilon\left(\frac{1-d}{2}\sin s + \frac{1}{2}(c-s)\cos s, (1-d)\cos s - (c-s)\sin s,\right. \\ &\left.\frac{\sqrt{3}(1-d)}{2}\sin s + \frac{\sqrt{3}}{2}(c-s)\cos s)\right) \end{aligned}$$

$$\begin{aligned}\widehat{\beta}'(s) &= \left(-\frac{1}{2}(c-s)\cos s,(c-s)\sin s,-\frac{\sqrt{3}}{2}(c-s)\cos s\right)\\ &\quad+\varepsilon\left(-\frac{1}{2}(c-s)\sin s-\frac{d}{2}\cos s,d\sin s-(c-s)\cos s,-\frac{\sqrt{3}d}{2}\cos s-\frac{\sqrt{3}}{2}(c-s)\sin s\right),\end{aligned}$$

The dual arc-length of the dual involute curve $\widehat{\beta}(s)$ from 0 to s is

$$\widetilde{\widetilde{s}}=\int_0^s\left\|\beta'(s)\right\|ds+\int_0^s\left\langle r_1,{\beta^*}'(s)\right\rangle=cs-\frac{s^2}{2}+\varepsilon sd,\frac{d\widetilde{\widetilde{s}}}{ds}=\lambda,$$

where $\left\|\widehat{\beta}'(s)\right\|=(c-s)+\varepsilon d=\lambda$. Frenet vectors and pole curve of the dual involute curve $\widehat{\beta}(s)$ can be calculated as follows.

From $R_1(s)=\frac{\widehat{\beta}'(s)}{\left\|\widehat{\beta}'(s)\right\|}$ we have

$$R_1(s)=\left(-\frac{1}{2}\cos s,\sin s,-\frac{\sqrt{3}}{2}\cos s\right)+\varepsilon\left(-\frac{1}{2}\sin s,-\cos s,-\frac{\sqrt{3}}{2}\sin s\right).$$

By $R_3=\frac{\widehat{\beta}'(s)\wedge\widehat{\beta}''(s)}{\left\|\widehat{\beta}'(s)\wedge\widehat{\beta}''(s)\right\|}$ we obtain

$$R_3(s)=\left(\frac{\sqrt{3}}{2},0,-\frac{1}{2}\right)+\varepsilon\left(\frac{2\sqrt{3}d}{c-s},0,\frac{2d}{c-s}\right).$$

Since $R_2=R_1\wedge R_3$ then we have

$$R_2(s)=\left(\frac{1}{2}\sin s,\cos s,\frac{\sqrt{3}}{2}\sin s\right)+\varepsilon\left(-\frac{1}{2}\cos s,\sin s,-\frac{\sqrt{3}}{2}\cos s\right).$$

The pole curve of the dual involute curve is $\overline{C}=\frac{QR_1+PR_3}{\sqrt{P^2+Q^2}}$. Since $P=\frac{\left\|\widehat{\beta}'(s)\wedge\widehat{\beta}''(s)\right\|}{\left\|\widehat{\beta}'(s)\right\|^3}=\frac{1}{\lambda}$ and $Q=\frac{\det\left(\widehat{\beta}'(s),\widehat{\beta}''(s),\widehat{\beta}'''(s)\right)}{\left\|\widehat{\beta}'(s)\wedge\widehat{\beta}''(s)\right\|^2}=0$ then we have

$$\overline{C}=\left(\frac{\sqrt{3}}{2},0,-\frac{1}{2}\right)+\varepsilon\left(\frac{2\sqrt{3}d}{c-s},0,\frac{2d}{c-s}\right).$$

In addition, the dual spherical areas limited by the dual ruled srfaces $[R_1],[R_2],[R_3]$ and $\left[\overline{C}\right]$ are as follow.

$$F_{R_1}=F_{R_2}=2\pi(1-n),$$

$$F_{R_3} = 2\pi\,(1-n) - \frac{1}{\lambda},$$

$$F_{\overline{C}} = 2\pi\,(1-n) + \ln(c-s) - \varepsilon\left(\frac{d}{c-s}\right) + c_1,\ \ c_1 \in \mathbb{R}.$$

The corresponding ruled surfaces have the following parametrizations, respectively

$$[R_1] = \left(-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + v\left(-\frac{1}{2}\cos s, \sin s, -\frac{\sqrt{3}}{2}\cos s\right)$$

$$[R_2] = \left(-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + v\left(\frac{1}{2}\sin s, \cos s, \frac{\sqrt{3}}{2}\sin s\right)$$

$$[R_3] = [\overline{C}] = \left(0, -\frac{2\sqrt{3}d}{c-s}, 0\right) + v\left(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right)$$

where $-5 \leq v \leq 5$ and $c = d = 10$. ($Figures$ $2, 3, 4$).

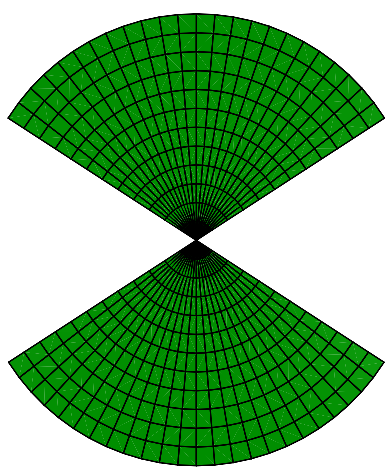

Figure 2: Ruled surface $[R_1]$ constructed by R_1 for s$\in [-1, 1]$

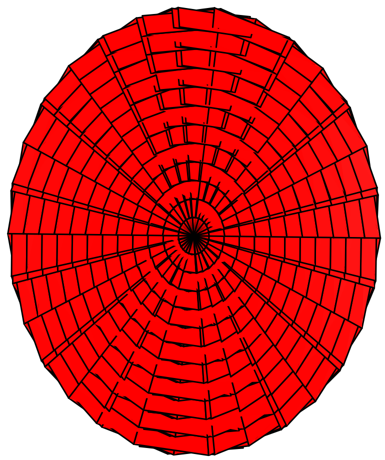

Figure 3: Ruled surface $[R_2]$ constructed by R_2 for $s \in [-5, 5]$

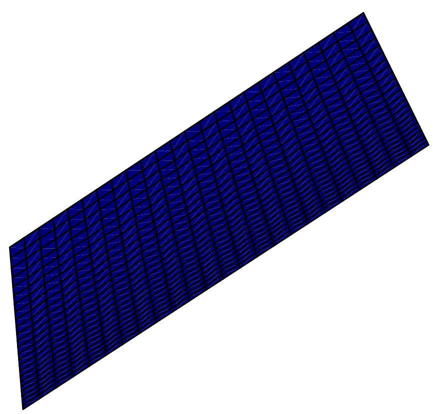

Figure 4: Ruled surface $[R_3] = \left[\overline{C}\right]$ constructed by $R_3(= \overline{C})$ for $s \in [-3, 3]$

5 Open Problem

In this paper, we investigated the dual geometric invariants of the closed ruled surface $\left[\overline{C}\right]$ corresponding to the pole curve $\overline{C}$ of the dual involute curve and some new interesting results about the developability of this surface were also given. Same study may be done Minkowski space, Galilean space or Heisenberg space. Another alternative is to generalize this work to higher dimensional spaces.

References

[1] M. Bilici, M. Çalışkan, *On the Involutes of the Spacelike Curve with a Timelike Binormal in Minkowski 3-Space*. Int. Math. Forum. 4(31), (2009), 1497-1509.

[2] M. Bilici, M. Çalışkan, *Some New Notes on the Involutes of the Timelike Curves in Minkowski 3-Space*. Int. J. Contemp. Math. Sci. 6(41), (2011), 2019-2030

[3] M. Bilici, M. Çalışkan, *A New Perspective on the Involutes of the Spacelike Curve with a Spacelike Binormal in Minkowski 3-Space*. J. Sci. Arts. 3(44), (2018), 573-582.

[4] M. Bilici, *On the invariants of ruled surfaces generated by the dual involute Frenet trihedron*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 66(2), (2017), 62-70.

[5] W. K. Clifford, *Preliminary sketch of biquaternions.* Proc. London Math. Soc. 4 (1873), 381-395.

[6] M. Çalışkan, M. Bilici, *Some Characterizations for the Pair of Involute-Evolute Curves in Euclidean Space E^3*. Bull. Pure Appl. Sci. 21(2), (2002), 289-294.

[7] H.W. Guggenheimer, *Differential Geometry*, McGraw-Hill, New York, (1963).

[8] O. Gürsoy, *The dual angle of pitch of a closed ruled surfaces*, Mech. and Mach. Theory. 25(2), (1990), 131-140.

[9] O. Gürsoy, *On the integral invariants of a closed ruled surface*, J. Geom. 25(2), (1990), 131-140.

[10] O. Gürsoy, *Some results on closed ruled surfaces and closed space curves*, Mech. and Mach. Theory. 27(3), (1992), 323-330.

[11] H. H. Hacısalihoğlu, *On the Pitch of a Closed Ruled Surface,* Mech. and Mach. Theory. 7(3), (1972) , 291–305.

[12] Ö. Köse, Ş. Nizamoğlu, M. Sezer, *An explicit characterization of dual spherical curves*, Doğa Turkish J. Math. 12, (1988), 105–113.

[13] H.R. Müller, *Kinematik Dersleri*, Ankara Üniversitesi Fen Fakültesi Yayınları, Mat. No. 2, Ankara, (1983).

[14] A. Sarioglugil, S. Senyurt, N. Kuruoglu, *On the integral invariants of the closed ruled surfaces generated by a parallel p-äquidistante dual centrode curves in the line space*, Hadronic J. 34 (3), (2011), 34-47.

[15] E. Study, *Geometrie der Dynamen*. Verlag Teubner, Leibzig, (1903).

[16] S. Şenyurt, *On Dual Ruled Surfaces*, Master Thesis, Ondokuz Mayıs University, Samsun (1994).

[17] S. Şenyurt, M. Bilici, M. & Calıskan, *Some characterizations for the involute curves in dual space*. Int. J. Math. Combin. 1(9), (2015). 113-125.

[18] Y. Yaylı, S. Saraçoğlu, *Ruled surfaces and dual spherical curves*, Acta Univ. Apulensis. 30, (2012), 337-354.

[19] A. Yücesan, N. Ayyildiz, A. C. & Çöken, *On Rectifying Dual Space Curves,* Rev. Mat. Complut. 20(2), (2007), 497-506.

[20] G. R. Veldkamp, *On the use of dual numbers, vectors, and matrices in instantaneous, spatial kinematics.* Mech. and Mach. Theory. 11, (1976), 141.156.