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A beautiful inequality chain for a triangle

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Abstract

We establish a new beautiful inequality chain involving the length elements of a triangle. After verifying by computer, we also propose some related interesting conjectures as open problems.

Keywords: *Triangle; Altitude; Angle-bisector; Medians; Symmedian; Radious of exeireles.*

1 Introduction and Main Result

For a given triangle ABC with side lengths a, b, c. Let m_a, m_b, m_c (corresponding to a, b, c, respectively) be the medians, h_a, h_b, h_c the altitudes, w_a, w_b, w_c the angle-bisectors, r_a, r_b, r_c the radii of excircles, and k_a, k_b, k_c the symmedians. Let R, r be its radius of circumcircle and radius of incircle, respectively.

From the monograph [2], we learn that L. Panaitopol established the following inequality (in 1982):

$$\frac{R}{r} \geq 2\frac{m_a}{h_a}. \tag{1.1}$$

The author of this paper pointed out several equivalent forms of Panaitopol's inequality in the recent paper [7] and gave a proof and application in the monographs [6, p.128-129]. In a Chinese book [1], B. Q. Liu conjectured that the following refinement of (1.1) holds:

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq 2\frac{m_a}{h_a}. \tag{1.2}$$

In [7], the author proved the double inequality (1.2) and the following extension:

$$2\frac{m_a}{h_a} \geq \frac{l_b}{l_c} + \frac{l_c}{l_b} \geq \frac{c+a}{a+b} + \frac{a+b}{c+a}. \tag{1.3}$$

where l_b, l_c are the corresponding latitudes of the triangle ABC or the angle-bisectors or the medians or the sysmmedians (l_a denotes similar meaning, we shall continuously use these three symbols in the sequel). In [7], the author also established some other similar inequality chains. For example,

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \geq 2\frac{m_b+m_c}{h_b+h_c} \geq \frac{c+a}{a+b} + \frac{a+b}{c+a}, \tag{1.4}$$

and

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \geq 2\frac{m_a+m_b+m_c}{h_a+h_b+h_c} \geq \frac{m_c+m_a}{m_a+m_b} + \frac{m_a+m_b}{m_c+m_a}. \tag{1.5}$$

Recently, the author also found some new similar results. For example,

$$\frac{b}{r_c} + \frac{c}{r_b} \geq \frac{b+c}{h_a} \geq \frac{b}{r_b} + \frac{c}{r_c} \geq \frac{b+c}{m_a}, \tag{1.6}$$

which can be easily proved.

Our main goal here is to establish the following beautiful chain:

Theorem 1.1 *In any triangle ABC, we have*

$$\frac{l_b}{r_b} + \frac{l_c}{r_c} \geq \frac{l_b+l_c}{h_a} \geq \frac{l_b}{r_c} + \frac{l_c}{r_b} \geq \frac{l_b+l_c}{m_a}. \tag{1.7}$$

In each case there is equality if and only if $b = c$.

As an application of Theorem 1.1, the following complete symmetric inequality chain can be obtained immediately:

$$2\sum\frac{l_a}{r_a} \geq \sum\frac{l_b+l_c}{h_a} \geq 2\sum\frac{l_a}{h_a} \geq \sum\frac{l_b+l_c}{m_a}, \tag{1.8}$$

where $\sum$ denote cyclic sums over the subscript (a, b, c).

In the last section, we shall propose some interesting conjectures related to inequality chain (1.7).

2 Proof of Theorem 1.1

2.1 Two Lemmas

Lemma 2.1 *In any triangle ABC, we have*

$$\frac{4s(b+c)(s-b)(s-c)}{(c+a)(a+b)} \le w_b w_c \le \frac{4sbca^2}{(b+c)(c+a)(a+b)}, \tag{2.1}$$

where $s = (a+b+c)/2$. Both equalities in (2.1) *hold if and only if $b = c$.*

Proof. Applying the following known angle-bisector formula:

$$w_a = \frac{2}{b+c}\sqrt{s(s-a)bc} \tag{2.2}$$

and the half-angle formula:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{2.3}$$

we obtain

$$w_b w_c = \frac{4abcs}{(c+a)(a+b)}\sin\frac{A}{2}. \tag{2.4}$$

In view of the well known inequality:

$$\sin\frac{A}{2} \le \frac{a}{b+c}, \tag{2.5}$$

the second inequality of (2.1) follows from (2.4) immediately. Also, it follows from (2.4) and (2.5) that

$$w_b w_c \ge \frac{4abcs}{(c+a)(a+b)}\sin^2\frac{A}{2}\cdot\frac{b+c}{a}.$$

Using (2.3) again, we get the first inequality of (2.1). Note that equality in (2.5) holds only when $b = c$, thus the equality condition of (2.1) is the same as that of (2.5). This completes the proof of Lemma 2.1.

Lemma 2.2 *In any triangle ABC, we have*

$$4m_b m_c \le 2a^2 + bc, \tag{2.6}$$

with equality if and only if $b = c$.

Inequality (2.6) can be obtained from the following identity:

$$16(m_b m_c)^2 = (2a^2+bc)^2 - 2(a+b+c)(b+c-a)(b-c)^2. \tag{2.7}$$

As far as I know, inequality (2.6) was first found by J. Chen (see [4]). Inequality (2.6) and the second inequality of (1.2) could be proved easily by applying Potlemy's inequality (cf. [3]).

In the following subsections, we shall prove the three inequalities of (1.7) in the proper order.

2.2 Proof of the first inequality of inequality chain (1.7)

In this section, we prove the first inequality of (1.7), that is

$$\frac{l_b}{r_b} + \frac{l_c}{r_c} \geq \frac{l_b + l_c}{h_a}. \tag{2.8}$$

This inequality actually includes four inequalities. We thus divide the proof into the following four cases.

Case 1. When l_b and l_c are the altitudes of the triangle ABC.

In this case, inequality (2.8) is that

$$\frac{h_b}{r_b} + \frac{h_c}{r_c} \geq \frac{h_b + h_c}{h_a}. \tag{2.9}$$

Using the following formulas:

$$h_a = \frac{2S}{a}, \tag{2.10}$$

$$r_a = \frac{S}{s-a}, \tag{2.11}$$

(where S is the area of ABC), it is easy to get identity:

$$\frac{h_b}{r_b} + \frac{h_c}{r_c} - \frac{h_b + h_c}{h_a} = \frac{(b-c)^2}{bc}, \tag{2.12}$$

which shows that inequality (2.9) is true and its equality holds if and only if $b = c$.

Case 2. When l_b and l_c are the angle-bisectors of the triangle ABC.

In this case, inequality (2.8) is that

$$\frac{w_b}{r_b} + \frac{w_c}{r_c} \geq \frac{w_b + w_c}{h_a}, \tag{2.13}$$

which is equivalent to the following after squaring both sides:

$$\frac{w_b^2}{r_b^2} + \frac{w_c^2}{r_c^2} + \frac{2w_b w_c}{r_b r_c} \geq \frac{w_b^2 + w_c^2 + 2w_b w_c}{h_a^2}.$$

According to Lemma 2.1, to prove the above inequality we only need to prove the following inequality:

$$A_0 \equiv \frac{w_b^2}{r_b^2} + \frac{w_c^2}{r_c^2} + \frac{2w_1}{r_b r_c} - \frac{w_b^2 + w_c^2 + 2w_2}{h_a^2} \geq 0, \tag{2.14}$$

where

$$w_1 = \frac{4s(b+c)(s-b)(s-c)}{(c+a)(a+b)}, \quad w_2 = \frac{4sbca^2}{(b+c)(c+a)(a+b)}.$$

With the help of software Maple, by using formulas (2.2), (2.10) and (2.11), we easily obtain the following identity:

$$A_0=\frac{4(b-c)^2A_1}{(b+c)(b+c-a)(c+a-b)(a+b-c)(c+a)^2(a+b)^2}, \tag{2.15}$$

where

$$\begin{aligned}A_1=&a^6+3(b+c)a^5+(b^2+3bc+c^2)a^4-(b+c)(b^2-4bc+c^2)a^3\\&+(b^2-bc+c^2)(b+c)^2a^2+(b+c)(b^2+bc+c^2)(b-c)^2a\\&+bc(b-c)^2(b+c)^2.\end{aligned}$$

Since

$$\begin{aligned}&(b^2+3bc+c^2)a^2-(b+c)(b^2-4bc+c^2)a+(b^2-bc+c^2)(b+c)^2\\&=(b+c)(b+c-a)(b^2-bc+c^2)+(b^2+3bc+c^2)a^2+3abc(b+c),\end{aligned} \tag{2.16}$$

we have $A_1>0$. Then $A_0\geq 0$ follows from (2.15) and inequality (2.13) is proved. Also, by identity (2.15) and Lemma 2.1, one deduces that equality in (2.1) holds if and only if $b=c$.

Case 3. When l_b and l_c are the medians of the triangle ABC.

In this case, inequality (2.8) is that

$$\frac{m_b}{r_b}+\frac{m_c}{r_c}\geq\frac{m_b+m_c}{h_a}, \tag{2.17}$$

which is equivalent to

$$\frac{m_b^2}{r_b^2}+\frac{m_c^2}{r_c^2}+\frac{2m_bm_c}{r_br_c}\geq\frac{m_b^2+m_c^2+2m_bm_c}{h_a^2}.$$

Now, by Lemma 2.2 we have

$$m_1\leq m_bm_c\leq m_2, \tag{2.18}$$

where

$$m_1=\frac{4m_b^2m_c^2}{2a^2+bc},\quad m_2=\frac{1}{4}(2a^2+bc).$$

Thus, we only need to prove the following inequality:

$$B_0\equiv\frac{m_b^2}{r_b^2}+\frac{m_c^2}{r_c^2}+\frac{2m_1}{r_br_c}-\frac{m_b^2+m_c^2+2m_2}{2h_a^2}\geq 0. \tag{2.19}$$

Again, using the formulas (2.10), (2.11), the median formula:

$$4m_a^2=2(b^2+c^2)-a^2, \tag{2.20}$$

and the Heron formula:

$$S=\sqrt{s(s-a)(s-b)(s-c)}, \tag{2.21}$$

we obtain the following identity:

$$B_0=\frac{(b-c)^2B_1}{16(2a^2+bc)S^2}, \tag{2.22}$$

where

$$B_1=4a^4+12(b+c)a^3-(6b^2+4bc+6c^2)a^2+6bc(b+c)a+(4b^2+9bc+4c^2)(b-c)^2.$$

We now set $s-a=x, s-b=2y, s-c=2z$, then $a=y+z, b=z+x, c=x+y$ $(x,y,z>0)$. Substituting them into the expression of B_1 gives

$$\begin{aligned}B_1=&12(y+z)x^3+(19y^2-30yz+19z^2)x^2+(y+z)(31y^2+6yz+31z^2)x\\&+(y+2z)(2y+z)(7y^2+10yz+7z^2).\end{aligned} \tag{2.23}$$

Note that $19y^2-30yz+19z^2>0$, thus the strict inequality $B_1>0$ holds and $B_0\geq 0$ follows from (2.22). Inequality (2.17) is proved and it is easy to know that its equality holds only when $b=c$.

Case 4. When l_b and l_c are the sysmmedians of the triangle ABC.

In this case, inequality (2.8) is that

$$\frac{k_b}{r_b}+\frac{k_c}{r_c}\geq\frac{k_b+k_c}{h_a}. \tag{2.24}$$

By the known symmedian formula (cf. [2, p.213] and [5]):

$$k_a=\frac{2bcm_a}{b^2+c^2}, \tag{2.25}$$

we obtain

$$k_bk_c=\frac{8bca^2m_bm_c}{(c^2+a^2)(a^2+b^2)}. \tag{2.26}$$

Then by Lemma 2.2, we have

$$k_1\leq k_bk_c\leq k_2, \tag{2.27}$$

where

$$k_1=\frac{16bca^2m_b^2m_c^2}{(2a^2+bc)(c^2+a^2)(a^2+b^2)},\quad k_2=\frac{bc(2a^2+bc)a^2}{(c^2+a^2)(a^2+b^2)}.$$

Therefore, in order to prove inequality (2.24) we only need to prove that

$$C_0\equiv\frac{k_b^2}{r_b^2}+\frac{k_c^2}{r_c^2}+\frac{2k_1}{r_br_c}-\frac{k_b^2+k_c^2+2k_2}{h_a^2}\geq 0. \tag{2.28}$$

Making use of formulas (2.10), (2.11), (2.20), (2.21) and software Maple, we obtain the following identity:

$$C_0 = \frac{a^2(b-c)^2C_1}{4(2a^2+bc)(c^2+a^2)^2(a^2+b^2)^2S^2}, \tag{2.29}$$

where

$$\begin{aligned} C_1 \equiv\ & 8(b+c)a^9 + (4b^2-4bc+4c^2)a^8 + 4(b+c)(2b^2+bc+2c^2)a^7 \\ & + (4b^4-10b^3c+4b^2c^2-10bc^3+4c^4)a^6 + 4(b+c)(b^2+4bc+c^2)bca^5 \\ & - 2(b^4+12b^2c^2+c^4)bca^4 + 4(b+c)^3b^2c^2a^3 \\ & + 2(2b^6-b^5c-5b^4c^2-5b^2c^4-bc^5+2c^6)bca^2 \\ & + 2(b+c)(b^2+c^2)b^3c^3a + 3(b-c)^2(b+c)^2b^3c^3. \end{aligned}$$

Putting $s-a=2x, s-b=y, s-c=z$, then $a=y+z, b=z+x, c=x+y$. Substituting them into C_1 and using Maple, we immediately obtain the following identity:

$$\begin{aligned} C_1 =\ & 8(y+z)x^9 + (32y^2+16yz+32z^2)x^8 + 4(y+z)(21y^2-2yz \\ & + 21z^2)x^7 + (151y^4+260y^3z+74y^2z^2+260yz^3+151z^4)x^6 \\ & + (y+z)(205y^4+440y^3z+38y^2z^2+440yz^3+205z^4)x^5 \\ & + (215y^6+971y^5z+1557y^4z^2+1458y^3z^3+1557y^2z^4 \\ & + 971yz^5+215z^6)x^4 + (y+z)(183y^6+896y^5z+1917y^4z^2 \\ & + 2104y^3z^3+1917y^2z^4+896yz^5+183z^6)x^3 + (118y^8 \\ & + 893y^7z+2955y^6z^2+5647y^5z^3+6886y^4z^4+5647y^3z^5 \\ & + 2955y^2z^6+893yz^7+118z^8)x^2 + (y+z)(68y^8+524y^7z \\ & + 1681y^6z^2+3236y^5z^3+3966y^4z^4+3236y^3z^5 \\ & + 1681y^2z^6+524yz^7+68z^8)x + (y+2z)(2y+z)(2y^2+3yz \\ & + 2z^2)(6y^4+13y^3z+22y^2z^2+13yz^3+6z^4)(y+z)^2. \end{aligned} \tag{2.30}$$

It is clear that $C_1 > 0$. Then inequality $C_0 \geq 0$ follows from (2.29) and inequality (2.24) is proved. Also, it is easily seen that equality in (2.24) holds if and only if $b=c$.

2.3 Proof of the second inequality of inequality chain (1.7)

In this section, we prove the second inequality of inequality chain (1.7):

$$\frac{l_b+l_c}{h_a} \geq \frac{l_b}{r_c} + \frac{l_c}{r_b}. \tag{2.31}$$

We also consider four cases as above to prove inequality (2.31).

Case 1. When l_b and l_c are the altitudes of the triangle ABC.
In this case, inequality (2.31) is that

$$\frac{h_b + h_c}{h_a} \geq \frac{h_b}{r_c} + \frac{h_c}{r_b}. \tag{2.32}$$

But it is easy to prove the following identity:

$$\frac{h_b + h_c}{h_a} - \frac{h_b}{r_c} - \frac{h_c}{r_b} = \frac{(b-c)^2}{bc}, \tag{2.33}$$

so we have inequality (2.32).

Case 2. When l_b and l_c are the angle-bisectors of the triangle ABC.
In this case, inequality (2.31) is that

$$\frac{w_b + w_c}{h_a} \geq \frac{w_b}{r_c} + \frac{w_c}{r_b}, \tag{2.34}$$

which is equivalent to

$$\frac{w_b^2 + w_c^2 + 2w_b w_c}{h_a^2} \geq \frac{w_b^2}{r_b^2} + \frac{w_c^2}{r_c^2} + \frac{2w_b w_c}{r_b r_c}.$$

Since we have the following simple known inequality:

$$r_b r_c \geq h_a^2, \tag{2.35}$$

it is suffice to show that

$$\frac{w_b^2 + w_c^2}{h_a^2} \geq \frac{w_b^2}{r_c^2} + \frac{w_c^2}{r_b^2}. \tag{2.36}$$

By using the previous formulas (2.2), (2.10) and (2.11), it is not difficult to obtain the following identity:

$$\frac{w_b^2 + w_c^2}{h_a^2} - \frac{w_b^2}{r_c^2} - \frac{w_c^2}{r_b^2} = \frac{4(b-c)^2 D_0}{(b+c-a)(c+a-b)(a+b-c)(c+a)^2(a+b)^2}, \tag{2.37}$$

where

$$D_0 = 2a^4 + (b+c)a^3 - (b^2 - 4bc + c^2)a^2 + (b+c)bca + (b-c)^2 bc.$$

Also, it is easy to get

$$\begin{aligned} D_0 =& 2(y+z)x^3 + (6y^2 + 8yz + 6z^2)x^2 + 2(y+z)(3y^2 + 5yz + 3z^2)x \\ &+ 2y^4 + 16y^3 z + 24y^2 z^2 + 16yz^3 + 2z^4, \end{aligned} \tag{2.38}$$

where $x = s - a, y = s - b, z = s - c$. So, we have $D_0 > 0$ and then inequality (2.36) follows from identity (2.37). Hence, inequality (2.34) is proved and its equality holds if and only if $b = c$.

Case 3. When l_b and l_c are the medians of the triangle ABC.

In this case, inequality (2.31) is that

$$\frac{m_b + m_c}{h_a} \geq \frac{m_b}{r_c} + \frac{m_c}{r_b}. \tag{2.39}$$

By Lemma 2.2, to prove this inequality we only need to prove the following inequality:

$$E_0 \equiv \frac{m_b^2 + m_c^2 + 2m_1}{h_a^2} - \frac{m_b^2}{r_c^2} - \frac{m_c^2}{r_b^2} - \frac{2m_2}{r_b r_c} \geq 0, \tag{2.40}$$

where m_1 and m_2 are the same as in (2.27). But it is easy to obtain

$$E_0 = \frac{(b-c)^2 E_1}{16(2a^2 + bc)S^2}, \tag{2.41}$$

where

$$E_1 = 4a^4 + (12b + 12c)a^3 + (-6b^2 - 4bc - 6c^2)a^2 + 6bc(b+c)a - bc(b-c)^2.$$

If we set $s - a = x, s - b = y, s - c = z$, then it is easy to get

$$\begin{aligned} E_1 =& (12y + 12z)x^3 + (y^2 + 6yz + z^2)x^2 + (y+z)(13y^2 + 42yz \\ &+ 13z^2)x + (y + 2z)(2y + z)(5y^2 + 14yz + 5z^2), \end{aligned} \tag{2.42}$$

so that $E_1 > 0$. And then $E_0 \geq 0$ follows from (2.41). Inequality (2.38) is proved and its equality holds only when $b = c$.

Case 4. When l_b and l_c are the symmedians of the triangle ABC.

In this case, inequality (2.31) is that

$$\frac{k_b + k_c}{h_a} \geq \frac{k_b}{r_c} + \frac{k_c}{r_b}. \tag{2.43}$$

By the previous inequality (2.27), we only need to prove the following inequality:

$$F_0 \equiv \frac{k_b^2 + k_c^2 + 2k_1}{h_a^2} - \frac{k_b^2}{r_c^2} - \frac{k_c^2}{r_b^2} - \frac{2k_2}{r_b r_c} \geq 0. \tag{2.44}$$

Making using of the previous formula (2.10), (2.11), (2.20), (2.25) and Maple, we obtain

$$F_0 \equiv \frac{a^2(b-c)^2 F_1}{4(2a^2 + bc)(c^2 + a^2)^2(a^2 + b^2)^2 S^2}, \tag{2.45}$$

where

$$\begin{aligned} F_1 =&8(b+c)a^9-(4b^2-12bc+4c^2)a^8+4(b+c)(2b^2+bc\\ &+2c^2)a^7-2(2b^2-bc+c^2)(b^2-bc+2c^2)a^6\\ &+4(b+c)(b^2+4bc+c^2)bca^5-2(3b^4+4b^3c+4bc^3\\ &+3c^4)bca^4+4(b+c)^3a^3b^2c^2+2(b^4-3b^3c\\ &-4b^2c^2-3bc^3+c^4)b^2c^2a^2+2(b+c)(b^2\\ &+c^2)b^3c^3a+(b-c)^2(b+c)^2b^3c^3. \end{aligned}$$

With the help of Maple, we immediately obtain the following identity:

$$\begin{aligned} F_1 =&8(y+z)x^9+(24y^2+32yz+24z^2)x^8+4(y+z)(13y^2+14yz\\ &+13z^2)x^7+(85y^4+236y^3z+254y^2z^2+236yz^3+85z^4)x^6\\ &+(y+z)(119y^4+368y^3z+354y^2z^2+368yz^3+119z^4)x^5\\ &+(109y^6+713y^5z+1639y^4z^2+2022y^3z^3+1639y^2z^4\\ &+713yz^5+109z^6)x^4+(y+z)(77y^6+712y^5z+1975y^4z^2\\ &+2568y^3z^3+1975y^2z^4+712yz^5+77z^6)x^3+(50y^8+607y^7z\\ &+2673y^6z^2+5925y^5z^3+7602y^4z^4+5925y^3z^5+2673y^2z^6\\ &+607yz^7+50z^8)x^2+(y+z)(44y^8+388y^7z+1547y^6z^2\\ &+3364y^5z^3+4298y^4z^4+3364y^3z^5+1547y^2z^6+388yz^7\\ &+44z^8)x+(y+2z)(2y+z)(2y^2+11yz+2z^2)(2y^2+3yz\\ &+2z^2)(y+z)^4, \end{aligned} \tag{2.46}$$

where $x = s-a, y = s-b, z = s-c$. It is clear that $F_1 > 0$. Then inequality $F_0 \geq 0$ follows from (2.45). This completes the proof of inequality (2.43). Also, it is easily seen that equality in (2.43) holds if and only if $b = c$.

2.4 Proof of the third inequality inequality chain of (1.7)

In this section, we prove the third inequality of inequality chain (1.7).

$$\frac{l_b}{r_c}+\frac{l_c}{r_b} \geq \frac{l_b+l_c}{m_a}. \tag{2.47}$$

We still consider the following four cases to finish the proof of (2.47).

Case 1. When l_b and l_c are the altitudes of the triangle ABC.

In this case, inequality is that

$$\frac{h_b}{r_c}+\frac{h_c}{r_b} \geq \frac{h_b+h_c}{m_a}. \tag{2.48}$$

Using the previous formulas (2.10), (2.11) and (2.20), we obtain the following identity:

$$\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)^2-\left(\frac{h_b+h_c}{m_a}\right)^2=\frac{(b-c)^2G_0}{4b^2c^2m_a^2}, \tag{2.49}$$

where

$$G_0=2(b+c)a^3-(b-c)^2a^2-4(b+c)(b^2+c^2)a+(b^2+3c^2)(3b^2+c^2).$$

Since G_0 can be rewritten as follows:

$$G_0=(b+c-a)^4+2a(b+c-a)^3+(b+c-a)^2(b-c)^2+(c+a-b)^2(a+b-c)^2, \tag{2.50}$$

so that $G_0>0$. Then inequality (2.48) follows from identity (2.49).

Case 2. When l_b and l_c are the angle-bisectors of the triangle ABC.

In this case, inequality (2.47) is that

$$\frac{w_b}{r_c}+\frac{w_c}{r_b}\geq\frac{w_b+w_c}{m_a}, \tag{2.51}$$

which is equivalent to

$$\frac{w_b^2}{w_c^2}+\frac{w_c^2}{r_b^2}+\frac{2w_bw_c}{r_br_c}\geq\frac{w_b^2+w_c^2+2w_bw_c}{m_a^2}.$$

In view of the known simple inequality:

$$m_a^2\geq r_br_c, \tag{2.52}$$

we only need to show that

$$\frac{w_b^2}{r_c^2}+\frac{w_c^2}{r_b^2}\geq\frac{w_b^2+w_c^2}{m_a^2}. \tag{2.53}$$

By applying the previous formulas (2.2), (2.10), (2.11) and (2.20), we can obtain the following identity:

$$\frac{w_b^2}{r_c^2}+\frac{w_c^2}{r_b^2}-\frac{w_b^2+w_c^2}{m_a^2}=\frac{a(b-c)^2H_0}{(b+c-a)(c+a)^2(a+b)^2m_a^2}. \tag{2.54}$$

where

$$H_0=2a^4+(b+c)a^3-3(b^2+c^2)a^2+bc(b+c)a+bc(3b^2+2bc+3c^2).$$

Note that H_0 can be written as follows:

$$\begin{aligned}H_0=&8x^4+18(y+z)x^3+4(y+2z)(2y+z)x^2+6zy(y+z)x\\&+2zy(5y^2+8zy+5z^2),\end{aligned} \tag{2.55}$$

where $x = s - a, y = s - b, z = s - c$. So, we have $H_0 > 0$ and (2.53) follows from (2.54). Inequality (2.51) is proved and one sees that its equality occurs only when $b = c$.

Case 3. When l_b and l_c are the medians of the triangle ABC.

In this case, inequality (2.47) is that

$$\frac{m_b}{r_c} + \frac{m_c}{r_b} \geq \frac{m_b + m_c}{m_a}. \tag{2.56}$$

By inequality (2.52), we only need to prove

$$\frac{m_b^2}{r_c^2} + \frac{m_c^2}{r_b^2} \geq \frac{m_b^2 + m_c^2}{m_a^2}. \tag{2.57}$$

Using (2.11), (2.20) and (2.21), we obtain

$$\frac{m_b^2}{r_c^2} + \frac{m_c^2}{r_b^2} - \frac{m_b^2 + m_c^2}{m_a^2} = \frac{(b-c)^2 I_0}{4m_a^2 S^2}, \tag{2.58}$$

where

$$\begin{aligned} I_0 =& -4a^4 + 6(b+c)a^3 + (11b^2 + 8bc + 11c^2)a^2 - 12(b+c)(b^2+c^2)a \\ &+ (b^2+c^2)(3b^2 + 2bc + 3c^2). \end{aligned}$$

Letting $s - a = x, s - b = y, s - c = z$, then $a = y + z, b = z + x, c = x + y$ and it is to obtain that

$$I_0 = \frac{1}{4} I_1, \tag{2.59}$$

where

$$\begin{aligned} I_1 =& 4x^4 - 4(y+z)x^3 - (3y^2 + 12yz + 3z^2)x^2 + 2(y+z)(y^2 + 5yz \\ &+ z^2)x + y^4 + 4y^3 z + 8y^2 z^2 + 4yz^3 + z^4. \end{aligned} \tag{2.60}$$

We are going to prove the following strict inequality:

$$I_1 > 0. \tag{2.61}$$

Without loss of generality we may assume that $y \geq z$. In this case, we can consider the following three cases (Case a, b and c) to finish the proof of (2.61).

Case a. The positive real numbers x, y, z satisfy $y \geq z \geq x$.

In this case, we may assume that $z = x + m, y = x + m + n$ $(m \geq 0, n \geq 0)$. Substituting them into the expression of I_1, we easily get

$$\begin{aligned} I_1 =& 24x^4 + (112m + 56n)x^3 + (174m^2 + 174mn + 41n^2)x^2 \\ &+ 10(2m+n)(5m^2 + 5mn + n^2)x + 18m^4 + 36m^3 n \\ &+ 26m^2 n^2 + 8mn^3 + n^4, \end{aligned} \tag{2.62}$$

Since $x > 0, m \geq 0$ and $n \geq 0$, one sees that $I_1 > 0$.

Case b. The positive real numbers x, y, z satisfy $y \geq x \geq z$.

In this case, we may assume that $x = z + m, y = z + m + n$ $(m \geq 0, n \geq 0)$ and it is easy to get

$$\begin{aligned} I_1 =& 24z^4 + (40m + 56n)z^3 + (17m^2 + 76mn + 41n^2)z^2 \\ &+ 2n(12m^2 + 21mn + 5n^2)z + n^2(n + 3m)^2, \end{aligned} \tag{2.63}$$

so that $I_1 > 0$.

Case c. The positive real numbers x, y, z satisfy $x \geq y \geq z$.

In this case, we may assume that $y = z + m, x = z + m + n$ $(m \geq 0, n \geq 0)$ and it is easy to get

$$\begin{aligned} I_1 =& (17z^2 - 24zn + 9n^2)m^2 + (40z^3 - 42z^2n - 6zn^2 + 12n^3)m \\ &+ 2(12z^4 - 8z^3n - 9z^2n^2 + 4zn^3 + 2n^4). \end{aligned} \tag{2.64}$$

We now set $m_0 = 12z^4 - 8z^3n - 9z^2n^2 + 4zn^3 + 2n^4$. Note that

$$m_0 = (7z - 6n)z^3 + (5z^2 + 8zn + 2n^2)(z - n)^2. \tag{2.65}$$

One sees that if $z > n$ then $m_0 > 0$. On the other hand, note that

$$m_0 = 2(n + 5z)(n - z)^3 + (22z^2 - 36zn + 15n^2)z^2. \tag{2.66}$$

We know again that if $n \geq z$ then $m_0 > 0$. In summary, we conclude that for any positive real numbers z and n inequality $m_0 > 0$ holds. Now, note that the left hand side of (2.64) is a quadratic function in m and we have $17z^2 - 24zn + 9n^2 > 0$ and $m_0 > 0$. Thus to prove $I_1 > 0$ we only need to show that its discriminant F_m is not greater than zero. A simple calculation gives

$$F_m = -4z(8z^3 + 8z^2n - 19zn^2 + 12n^3)(z - n)^2. \tag{2.67}$$

Since $8z^2n - 19zn^2 + 12n^3 = (8z^2 - 19zn + 12n^2)z > 0$, so we have $F_m \leq 0$. Hence, we have proved $I_1 > 0$ under the Case c.

Combining the arguments of Case a, b and c, we conclude that inequality (2.61) holds for any positive real numbers x, y, z. Hence, the strict inequality $I_0 > 0$ from the relation (2.59). And we further deduce that inequality (2.57) holds from (2.58). Inequality (2.56) is proved and it is clear that its equality holds only when $b = c$.

Case 4. When l_b and l_c are the symmedians of the triangle ABC.

In this case, inequality (2.47) is that

$$\frac{k_b}{r_c} + \frac{k_c}{r_b} \geq \frac{k_b + k_c}{m_a}. \tag{2.68}$$

By the above inequality (2.52), to prove (2.68) we only need to show that

$$\frac{k_b^2}{r_c^2}+\frac{k_c^2}{r_b^2}\geq\frac{k_b^2+k_c^2}{m_a^2}. \tag{2.69}$$

Making use of (2.11), (2.20) and (2.25) and software Maple, we obtain

$$\frac{k_b^2}{r_c^2}+\frac{k_c^2}{r_b^2}-\frac{k_b^2+k_c^2}{m_a^2}=\frac{a^2(b-c)^2J_0}{m_a^2S^2(c^2+a^2)^2(a^2+b^2)^2}, \tag{2.70}$$

where

$$\begin{aligned}J_0=&4(b+c)a^9-2(b^2+c^2)a^8-4(b+c)(b^2+c^2)a^7+(4b^4+4b^3c\\&+6b^2c^2+4bc^3+4c^4)a^6-8(b+c)(b^2+bc+c^2)(b^2-bc+c^2)a^5\\&+2(b^2+bc+c^2)(3b^4-b^3c+8b^2c^2-bc^3+3c^4)a^4\\&-14(b+c)(b^2+c^2)b^2c^2a^3+(13b^4+8b^3c+20b^2c^2\\&+8bc^3+13c^4)b^2c^2a^2-4(b+c)(b^2+c^2)^2b^2c^2a\\&-(3b^2+2bc+3c^2)(b^4-4b^2c^2+c^4)b^2c^2.\end{aligned}$$

Putting $s-a=x, s-b=y, s-c=z$, then we have $a=y+z, b=z+x, c=x+y$. With the help of Maple, we immediately get

$$J_0=\frac{1}{4}J_1, \tag{2.71}$$

where

$$\begin{aligned}J_1=&4x^{10}+12(y+z)x^9+(13y^2+72yz+13z^2)x^8-2(y+z)(5y^2-82yz\\&+5z^2)x^7+(-31y^4+168y^3z+462y^2z^2+168yz^3-31z^4)x^6\\&-2(y+z)(10y^4-65y^3z-254y^2z^2-65yz^3+10z^4)x^5\\&+(12y^6+88y^5z+486y^4z^2+824y^3z^3+486y^2z^4+88yz^5\\&+12z^6)x^4+(8y^6+44y^5z+105y^4z^2+136y^3z^3+105y^2z^4\\&+44yz^5+8z^6)y^2z^2+2(y+z)(8y^6+22y^5z+85y^4z^2+146y^3z^3\\&+85y^2z^4+22yz^5+8z^6)x^3-2(y+z)(4y^6+14y^5z+5y^4z^2\\&+4y^3z^3+5y^2z^4+14yz^5+4z^6)xyz+(4y^8-8y^7z-47y^6z^2\\&-40y^5z^3-30y^4z^4-40y^3z^5-47y^2z^6-8yz^7+4z^8)x^2.\end{aligned}$$

We can use the method of proving (2.61) to prove inequality $J_1>0$. In the above Case a, we may assume that $z=x+m, y=x+m+n (m\geq 0, n\geq 0)$. Substituting them into J_1 and expanding out by making use of Maple, we find that all of the terms (for x, m, n) are non-negative. Hence, inequality $J_1>0$ holds under Case a. Similarly, we know that $J_1>0$ is valid under the above

Case b and Case c. Therefore, we conclude that $J_1 > 0$ holds for any positive real numbers x, y, z.

Finally, we deduce that $J_0 \geq 0$ by relation (2.71) and inequality (2.69) follows from (2.70). Inequality (2.68) is proved and it is clear that its equality holds if and only if $b = c$. So far, we complete the proof of Theorem 1.1.

3 Some Open Problems

In [7], the author considered exponential generalizations of the second inequality of (1.3) and conjectured that for any positive real number k the follow inequality holds:

$$\frac{l_b^k}{l_c^k} \geq \frac{c^k + a^k}{a^k + b^k} + \frac{a^k + b^k}{c^k + a^k}. \tag{3.1}$$

This is still an open problem.

For inequality chain (1.7), we propose here some conjectures as new open problems.

Conjecture 3.1 *If a real number k satisfies $k > 1$ or $k \leq -1$, then for any triangle ABC the following inequality holds:*

$$\frac{l_b^k}{r_b} + \frac{l_c^k}{r_c} \geq \frac{l_b^k + l_c^k}{h_a} \geq \frac{l_b^k}{r_c} + \frac{l_c^k}{r_b}. \tag{3.2}$$

Conjecture 3.2 *Let p and q be two real numbers such that $p > 0, q \geq 1$ or $0 < p \leq 1, q < 0$, then for any triangle ABC the following inequality holds:*

$$\frac{l_b^p}{r_b^q} + \frac{l_c^p}{r_c^q} \geq \frac{l_b^p + l_c^p}{h_a^q}. \tag{3.3}$$

If $0 < q \leq 1, p < 0$, then the inequality holds reversely.

Conjecture 3.3 *Let p and q be two real numbers such that $0 < p \leq 1, 0 < q \leq 1$ or $p \geq q > 0, q \leq 1$, then for any triangle ABC the following inequality holds:*

$$\frac{l_b^p + l_c^p}{h_a^q} \geq \frac{l_b^p}{r_c^q} + \frac{l_c^p}{r_b^q} \tag{3.4}$$

If $p > 0 > q$ or $p < 0, q \geq 1$, then the above inequality holds reversely.

Conjecture 3.4 *Let p and q be two real numbers such that $q \geq 1 \geq p$, then for any triangle ABC the following inequality holds:*

$$\frac{l_b^p}{r_c^q} + \frac{l_c^p}{r_b^q} \geq \frac{l_b^p + l_c^p}{m_a^q}. \tag{3.5}$$

Conjecture 3.5 *Let p and q be two real numbers such that $p > 0, q \geq p+2$ or $q > 0 > p$, then for any triangle ABC the following inequality holds:*

$$\frac{m_b^p}{r_c^q} + \frac{m_c^p}{r_b^q} \geq \frac{m_b^p + m_c^p}{m_a^q}. \tag{3.6}$$

Conjecture 3.6 *Let p and q be two real numbers such that $q > 0, q \geq p$, then for any triangle ABC the following inequality holds:*

$$\frac{w_b^p}{r_c^q} + \frac{w_c^p}{r_b^q} \geq \frac{w_b^p + w_c^p}{m_a^q}. \tag{3.7}$$

If $p \leq q < 0$, then the inequality holds reversely. If $p > 0 > q$, then the inequality holds for the acute triangle ABC. If $p \geq 2, 0 < q \leq 1$ or $8 \geq p \geq q+1, q > 0$, then the inequality holds reversely for the acute triangle ABC.

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