Int. J. Open Problems Compt. Math., Vol. 15, No. 1, March 2022
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Power Semigroups on Semihypergroups

Kanruethai Jeenkaew, Sorasak Leeratanavalee

Ph.D. Degree Program in Mathematics, Department of Mathematics,
Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand
e-mail:kanruethai_jeenkaew@cmu.ac.th

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai, 50200, Thailand
e-mail:sorasak.l@cmu.ac.th

Abstract

Based on the development of fuzzy mathematics, all kinds of algebraic structure are upgraded from its universes to its power sets. In this paper, we construct a new structure which is called a power semigroup on a semihypergroup and then study some of its algebraic properties.

Keywords: *Semihypergroups, Semigroups, Power Semigroups.*
2010 Mathematics Subject Classification: 20M10, 20M15.

1 Introduction

The notion of hypergroups was first introduced by Marty [22]. There are a lot of results on hypergroups have been derived (see [2, 5, 12]). In 2010, Shengmin Ma, Honghi Mi and Lujing Huo introduced the concept of power groups on hypergroups and discussed on its character and its structure [21]. Algebraic hyperstructures are generalizations of classical algebraic structures. A direction of the study on hyperstructures is the theory of semihypergroups which was introduced by Marty in 1934 [22]. In the presented paper, we use the similar idea of Shengmin Ma et al. to construct power semigroups on semihypergroups and study some of its algebraic properties.

2 Preliminaries

In this section, we first recall some definitions, examples which will be used throughout this paper.

Definition 2.1 *[5] Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H. A mapping $\circ : H \times H \to P^*(H)$ is called a hyperoperation on H. Then $(H, \circ)$ is called a hypergroupoid.*

Shengmin Ma et al. defined a mapping $\cdot : P^*(H) \times P^*(H) \to P^*(H)$ as follows.

For any $A, B \in P^*(H)$, $A \cdot B = \bigcup\{a \circ b \mid a \in A, b \in B\}$. In case of $A = \{a\}$ (or $B = \{b\}$), we write $a \cdot B$ (or $A \cdot b$) instead of $\{a\} \cdot B$ (or $A \cdot \{b\}$). In case of $A = \emptyset$ (or $B = \emptyset$), $\emptyset \cdot B = \emptyset$ (or $A \cdot \emptyset = \emptyset$).

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for every $x, y, z \in H$, $(x \circ y) \cdot z = x \cdot (y \circ z)$, i.e. $\bigcup\{u \circ z \mid u \in x \circ y\} = \bigcup\{x \circ v \mid v \in y \circ z\}$.

Example 2.2 *Let $H = \{a, b, c\}$ and $\circ$ be a hyperoperation defined by the following table.*

$\circ$	a	b	c
a	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{a, b\}$	H
c	$\{c\}$	H	H

Then $(H, \circ)$ is a semihypergroup.

Example 2.3 *[8] Let $H = \{x, y, z, t\}$ and $\circ$ be a hyperoperation defined by the following table.*

$\circ$	x	y	z	t
x	$\{x\}$	$\{x, y\}$	$\{x, z\}$	H
y	$\{y\}$	$\{y\}$	$\{y, t\}$	$\{y, t\}$
z	$\{z\}$	$\{z, t\}$	$\{z\}$	$\{z, t\}$
t	$\{t\}$	$\{t\}$	$\{t\}$	$\{t\}$

Then $(H, \circ)$ is a semihypergroup.

Example 2.4 *[7] Let $H = [0, 1]$ and $\circ$ be a hyperoperation defined by $x \circ y = [0, xy]$ for any $x, y \in [0, 1]$. Then $(H, \circ)$ is a semihypergroup. Moreover, let $t \in [0, 1]$ and $T = [0, t]$. Then $(T, \circ)$ is a semihypergroup.*

3 Power semigroups on semihypergroups

In this section, we construct a new structure which is called a power semigroup on a semihypergroup and then study some of its properties.

Definition 3.1 *Let $(H, \circ)$ be a semihypergroup and $\emptyset \neq \mathcal{S} \subseteq P^*(H)$. Defined $\cdot : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ as in Definition 2.1. It is obviously that $(\mathcal{S}, \cdot)$ is a semigroup. We call $(\mathcal{S}, \cdot)$, a power semigroup on a semihypergroup $(H, \circ)$.*

Example 3.2 *Let $H = \{a, b, c\}$ be a semihypergroup with a hyperoperation $\circ$ defined by the following table.*

$\circ$	a	b	c
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$
c	$\{a, c\}$	$\{b, c\}$	$\{c\}$

Let $\mathcal{S} = \{\{a\}, \{a, b\}\}$ be a subset of $P^*(H)$. Define $\cdot$ as in Definition 3.1, then we have the following table.

$\cdot$	$\{a\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

It is obviously that $\mathcal{S}$ is a power semigroup on the semihypergroup H.

Let $\mathcal{T} = \{\{a\}, \{b, c\}\}$ be a subset of $P^*(H)$. Define $\cdot$ as in Definition 3.1, then we have the following table.

$\cdot$	$\{a\}$	$\{b, c\}$
$\{a\}$	$\{a\}$	$\{a, b, c\}$
$\{b, c\}$	$\{a, b, c\}$	$\{b, c\}$

We conclude that $\mathcal{T}$ is not a power semigroup on the semihypergroup H.

Next, we study on some algebraic properties of a power semigroup on a given semihypergroup

Definition 3.3 *Let $(H, \circ)$ be a semihypergroup and $(\mathcal{S}, \cdot)$ be a power semigroup on the semihypergroup H. A nonempty subset $\mathcal{N}$ of $\mathcal{S}$ is called a subsemigroup of $\mathcal{S}$ if $A \cdot B \in \mathcal{N}$ for any $A, B \in \mathcal{N}$.*

Example 3.4 *From Example 3.2, let* $\mathcal{N} = \{\{a, b\}\}$ *be a nonempty subset of* $\mathcal{S}$. *We have* $\{a, b\} \cdot \{a, b\} = \{a, b\} \in \mathcal{N}$. *Then* $\mathcal{N}$ *is a subsemigroup of* $\mathcal{S}$.

Proposition 3.5 *Let* $(\mathcal{S}, \cdot)$ *be a power semigroup on the semihypergroup* H. *If* $\{\mathcal{N}_\alpha \mid \alpha \in I$ *where* I *is an index set*$\}$ *is any nonempty collection of subsemigroups of* $\mathcal{S}$ *and* $\bigcap_{\alpha \in I} \mathcal{N}_\alpha \neq \emptyset$ *then* $\bigcap_{\alpha \in I} \mathcal{N}_\alpha$ *is a subsemigroup of* $\mathcal{S}$.

Proof Let $\{\mathcal{N}_\alpha \mid \alpha \in I$ where I is an index set$\}$ be any nonempty collection of subsemigroups of $\mathcal{S}$ and $\bigcap_{\alpha \in I} \mathcal{N}_\alpha \neq \emptyset$. Then $\mathcal{N}_\alpha \subseteq \mathcal{S}$ for all $\alpha \in I$. Hence $\bigcap_{\alpha \in I} \mathcal{N}_\alpha \subseteq \mathcal{N}_\alpha \subseteq \mathcal{S}$. Let $X, Y \in \bigcap_{\alpha \in I} \mathcal{N}_\alpha$. Then $X, Y \in \mathcal{N}_\alpha$ for all $\alpha \in I$. So $X \cdot Y \in \mathcal{N}_\alpha$ for all $\alpha \in I$. Hence $X \cdot Y \in \bigcap_{\alpha \in I} \mathcal{N}_\alpha$. Therefore $\bigcap_{\alpha \in I} \mathcal{N}_\alpha$ is a subsemigroup of $\mathcal{S}$.

Remark For the union, $\bigcup_{\alpha \in I} \mathcal{N}_\alpha$ is not necessary a subsemigroup of $\mathcal{S}$ as in the following example.

Example 3.6 *From Example 3.2, let* $\mathcal{S}' = \{\{a\}, \{b, c\}, \{a, b, c\}\}$ *be a subset of* $P^*(H)$. *Define* $\cdot$ *as in Definition 3.1, then we have the following table.*

$\cdot$	$\{a\}$	$\{b, c\}$	$\{a, b, c\}$
$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

It is obviously that $\mathcal{S}'$ *is a power semigroup on the semihypergroup* H. *We have* $\{\{a\}\}, \{\{b, c\}\}, \{\{a, b, c\}\}, \{\{a\}, \{a, b, c\}\}, \{\{b, c\}, \{a, b, c\}\}$ *and* $\{\{a\}, \{b, c\}, \{a, b, c\}\}$ *are subsemigroups of* $\mathcal{S}'$ *but* $\{\{a\}\} \cup \{\{b, c\}\} = \{\{a\}, \{b, c\}\}$ *is not a subsemigroup of* $\mathcal{S}'$.

Definition 3.7 *Let* $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ *be power semigroups on semihypergroups* $(H, \circ)$ *and* $(H', \circ')$ *respectively. A mapping* $h : \mathcal{A} \to \mathcal{B}$ *is called a homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$ *if* $h(X \cdot Y) = h(X) \cdot' h(Y)$ *for any* $X, Y \in \mathcal{A}$. *A homomorphism* h *is called a monomorphism, epimorphism, or isomorphism if* h *is one-to-one, onto, or a bijection, respectively. A homomorphism* $h : \mathcal{A} \to \mathcal{A}$ *is called an endomorphism and an isomorphism* $h : \mathcal{A} \to \mathcal{A}$ *is called an automorphism.*

Theorem 3.8 *Let* $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ *be power semigroups on semihypergroups* $(H, \circ)$ *and* $(H', \circ')$ *respectively. Let* $h : \mathcal{A} \to \mathcal{B}$ *be a homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$.

(i) If $\mathcal{A}$ *is commutative then* $h(\mathcal{A})$ *is commutative.*

(ii) If h *is onto and* $\mathcal{A}$ *is commutative then* $\mathcal{B}$ *is commutative.*

Proof (i) Let $\mathcal{A}$ be commutative and $X', Y' \in h(\mathcal{A})$. Then there exist $X, Y \in \mathcal{A}$ such that $h(X) = X'$ and $h(Y) = Y'$. Since $\mathcal{A}$ is commutative, $X \cdot Y = Y \cdot X$. Hence $X' \cdot' Y' = h(X) \cdot' h(Y) = h(X \cdot Y) = h(Y \cdot X) = h(Y) \cdot' h(X) = Y' \cdot' X'$. Then $h(\mathcal{A})$ is commutative.
(ii) It follows from (i).

Theorem 3.9 *Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ and $(\mathcal{C}, \cdot'')$ be power semigroups on semihypergroups $(H, \circ)$, $(H', \circ')$ and $(H'', \circ'')$ respectively. If $h : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ are homomorphisms then $gh : \mathcal{A} \to \mathcal{C}$ is also a homomorphism. (the notation gh means the composition of g and h)*

Proof Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ and $(\mathcal{C}, \cdot'')$ be power semigroups on semihypergroups $(H, \circ)$, $(H', \circ')$ and $(H'', \circ'')$ respectively and $h : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ be homomorphisms. Then $gh : \mathcal{A} \to \mathcal{C}$. Let $X, Y \in \mathcal{A}$. Then $(gh)(X \cdot Y) = g(h(X \cdot Y)) = g(h(X) \cdot' h(Y)) = g(h(X)) \cdot'' g(h(Y)) = (gh)(X) \cdot'' (gh)(Y)$. Therefore $gh : \mathcal{A} \to \mathcal{C}$ is a homomorphisms.

Corollary 3.10 *The composition of monomorphisms is a monomorphism, the composition of epimorphisms is an epimorphism, the composition of isomorphisms is an isomorphism, and the composition of automorphisms is an automorphism.*

Theorem 3.11 *Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$.*

(i) If $\mathcal{N}$ is a subsemigroup of $\mathcal{A}$ then $h(\mathcal{N})$ is a subsemigroup of $\mathcal{B}$.

(ii) If $\mathcal{L}$ is a subsemigroup of $\mathcal{B}$ and $h^{-1}(\mathcal{L}) \neq \emptyset$ then $h^{-1}(\mathcal{L})$ is a subsemigroup of $\mathcal{A}$ where $h^{-1}(\mathcal{L}) = \{X \in \mathcal{A} \mid h(X) \in \mathcal{L}\}$.

Proof (i) Let $\mathcal{N}$ be a subsemigroup of $\mathcal{A}$. We can see that $h(\mathcal{N}) \subseteq \mathcal{B}$ and since $\mathcal{N}$ is a subsemigroup of $\mathcal{A}$, $\mathcal{N}$ is not empty. For any $N \in \mathcal{N}$, $h(N) \in h(\mathcal{N})$. So $h(\mathcal{N})$ is not empty. Let $X', Y' \in h(\mathcal{N})$. Then there exist $X, Y \in \mathcal{N}$ such that $h(X) = X'$ and $h(Y) = Y'$. Hence $X' \cdot' Y' = h(X) \cdot' h(Y) = h(X \cdot Y)$. Since $X, Y \in \mathcal{N}$ and $\mathcal{N}$ is a subsemigroup of $\mathcal{A}$, $X \cdot Y \in \mathcal{N}$. Then $X' \cdot' Y' = h(X \cdot Y) \in h(\mathcal{N})$. Therefore $h(\mathcal{N})$ is a subsemigroup of $\mathcal{B}$.
(ii) Let $\mathcal{L}$ be a subsemigroup of $\mathcal{B}$ and $h^{-1}(\mathcal{L}) \neq \emptyset$. Let $X, Y \in h^{-1}(\mathcal{L})$. By the definition of $h^{-1}(\mathcal{L})$, we have $h^{-1}(\mathcal{L}) \subseteq \mathcal{A}$ and since $\mathcal{L}$ is a subsemigroup of $\mathcal{B}$, $\mathcal{L}$ is not empty. Then $h(X), h(Y) \in \mathcal{L}$. Hence $h(X \cdot Y) = h(X) \cdot' h(Y) \in \mathcal{L}$. Then $X \cdot Y \in h^{-1}(\mathcal{L})$. Therefore $h(\mathcal{N})$ is a subsemigroup of $\mathcal{B}$.

Definition 3.12 *Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Define $kerh = \{(X, Y) \in \mathcal{A} \times \mathcal{A} \mid h(X) = h(Y)\}$ as a relation on $\mathcal{A}$. We call $kerh$, the kernel of $\mathcal{A}$.*

Theorem 3.13 *Let* $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ *be power semigroups on semihypergroups* $(H, \circ)$ *and* $(H', \circ')$ *respectively. Let* $h : \mathcal{A} \to \mathcal{B}$ *be a homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$*. Then* $kerh$ *is a congruence on* $\mathcal{A}$*.*

Proof By Definition 3.12, we can see that $kerh$ is reflexive, symmetric and transitive. Let $(X, Y), (M, N) \in kerh$. Then $h(X) = h(Y)$ and $h(M) = h(N)$. We have $h(X \cdot M) = h(X) \cdot' h(M) = h(Y) \cdot' h(N) = h(Y \cdot N)$. That is $(X \cdot M, Y \cdot N) \in kerh$. Similarly, we can prove that $(M \cdot X, N \cdot Y) \in kerh$. So $kerh$ is a congruence on $\mathcal{A}$.

For any $A \in \mathcal{A}$, we define the equivalence class of A modulo $kerh$ by $A/kerh = \{B \in \mathcal{A} \mid (B, A) \in kerh\}$ and $\{A/kerh \mid A \in \mathcal{A}\}$ denoted by $\mathcal{A}/kerh$.

Theorem 3.14 *Let* $(\mathcal{A}, \cdot)$ *be a power semigroup on a semihypergroup* $(H, \circ)$ *and* θ *be a congruence on* $\mathcal{A}$*. Then we have* $(\mathcal{A}/\theta, \star)$ *is an algebra such that* $\star : (\mathcal{A}/\theta)^2 \to \mathcal{A}/\theta$ *is defined by* $X/\theta \star Y/\theta = (X \cdot Y)/\theta$*. Moreover,* $(\mathcal{A}/\theta, \star)$ *is also a semigroup.*

Proof We will show that $\star$ is a binary operation on $\mathcal{A}/\theta$. Let $(X_1/\theta, Y_1/\theta), (X_2/\theta, Y_2/\theta) \in (\mathcal{A}/\theta)^2$ such that $(X_1/\theta, Y_1/\theta) = (X_2/\theta, Y_2/\theta)$. So $X_1/\theta = X_2/\theta$ and $Y_1/\theta = Y_2/\theta$. Hence $(X_1, X_2) \in \theta$ and $(Y_1, Y_2) \in \theta$. Since θ is a congruence on $\mathcal{A}$, $(X_1 \cdot Y_1, X_2 \cdot Y_2) \in \theta$. Then $(X_1 \cdot Y_1)/\theta = (X_2 \cdot Y_2)/\theta$. That is $X_1/\theta \star Y_1/\theta = X_2/\theta \star Y_2/\theta$. Therefore $\star$ is a binary operation on $\mathcal{A}/\theta$. Next, we will show that $(\mathcal{A}/\theta, \star)$ is associative. Let $X/\theta, Y/\theta, Z/\theta \in \mathcal{A}/\theta$. Then $(X/\theta \star Y/\theta) \star Z/\theta = (X \cdot Y)/\theta \star Z/\theta = ((X \cdot Y) \cdot Z)/\theta = (X \cdot (Y \cdot Z))/\theta = X/\theta \star (Y \cdot Z)/\theta = X/\theta \star (Y/\theta \star Z/\theta)$. Then $(\mathcal{A}/\theta, \star)$ is associative.

We call $(\mathcal{A}/\theta, \star)$, the quotient algebra of $(\mathcal{A}, \cdot)$.

Theorem 3.15 *Let* $(\mathcal{A}, \cdot)$ *be a power semigroup on a semihypergroup* $(H, \circ)$ *and* θ *be a congruence on* $\mathcal{A}$*. Then there exists a surjective homomorphism* $g : \mathcal{A} \to \mathcal{A}/\theta$*.*

Proof Let $(\mathcal{A}, \cdot)$ be a power semigroup on a semihypergroup $(H, \circ)$ and θ be a congruence on $\mathcal{A}$. By Theorem 3.14, $(\mathcal{A}/\theta, \star)$ is an algebra. Define $g : \mathcal{A} \to \mathcal{A}/\theta$ by $g(A) = A/\theta$ for any $A \in \mathcal{A}$. (well-defined) Let $A_1, A_2 \in \mathcal{A}$ such that $A_1 = A_2$. Hence $A_1/\theta = A_2/\theta$. Then $g(A_1) = A_1/\theta = A_2/\theta = g(A_2)$. So g is well-defined. Let $A_1, A_2 \in \mathcal{A}$. Then $g(A_1 \cdot A_2) = (A_1 \cdot A_2)/\theta = A_1/\theta \star A_2/\theta = g(A_1) \star g(A_2)$. Hence g is a homomorphism. Let $A/\theta \in \mathcal{A}/\theta$. There is $A \in \mathcal{A}$ such that $g(A) = A/\theta$. So g is a surjective homomorphism.

We call $g : \mathcal{A} \to \mathcal{A}/\theta$ in Theorem 3.15, the natural homomorphism and g is denoted by nat_θ.

Theorem 3.16 *Let* $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$, $(\mathcal{C}, \cdot'')$ *be power semigroups on semihypergroups* $(H, \circ)$, $(H', \circ')$ *and* $(H'', \circ'')$ *respectively. Let* $h : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ *be two homomorphisms such that* g *is surjective. Then* *(i) There exists a homomorphism $f : \mathcal{C} \to \mathcal{B}$ such that $fg = h$ if and only if $kerg \subseteq kerh$.*

(ii) The function f in (1) is unique.

(iii) The function f in (1) is injective if and only if $kerg = kerh$.

(iv) The function f in (1) is surjective if and only if h is surjective.

Proof Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$, $(\mathcal{C}, \cdot'')$ be power semigroups on semihypergroups $(H, \circ)$, $(H', \circ')$ and $(H'', \circ'')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ be two homomorphisms such that g is surjective. (i)($\Rightarrow$) Let there exists a homomorphism $f : \mathcal{C} \to \mathcal{B}$ such that $fg = h$ and $(X, Y) \in kerg$. Then $g(X) = g(Y)$. Hence $f(g(X)) = f(g(Y))$. So $h(X) = fg(X) = f(g(X)) = f(g(Y)) = fg(Y) = h(Y)$. Then $(X, Y) \in kerh$.
($\Leftarrow$) Let $kerg \subseteq kerh$ and for any $C \in \mathcal{C}$, $f(C) = (hg^{-1})(C)$. We will show that for any $C \in \mathcal{C}$, $f(C)$ gave only one value. Let $C \in \mathcal{C}$ and assume $A_1, A_2 \in g^{-1}(C) = \{X \in \mathcal{A} \mid g(X) = C\}$. So $g(A_1) = C = g(A_2)$. Hence $(A_1, A_2) \in kerg \subseteq kerh$. So $h(A_1) = h(A_2)$. Consider $f(C) = (hg^{-1})(C) = h(g^{-1}(C))$. Since $h(A_1) = h(A_2)$, $h(g^{-1}(C))$ gave only one value for $A_1, A_2 \in g^{-1}(C)$. Then for any $C \in \mathcal{C}$, $f(C)$ gave only one value. So $fg = h$. We will show that f is a homomorphism. Let $C_1, C_2 \in \mathcal{C}$. Since g is surjective, there exists $A_1, A_2 \in \mathcal{A}$ such that $g(A_1) = C_1$ and $g(A_2) = C_2$. Then $f(C_1 \cdot'' C_2) = f(g(A_1) \cdot'' g(A_2)) = f(g(A_1 \cdot A_2)) = fg(A_1 \cdot A_2) = h(A_1 \cdot A_2) = h(A_1) \cdot' h(A_2) = fg(A_1) \cdot' fg(A_2) = f(g(A_1)) \cdot' f(g(A_2)) = f(C_1) \cdot' f(C_2)$.
(ii) Let $f' : \mathcal{C} \to \mathcal{B}$ be a function such that $f'g = h$. We will show that $f' = f$. Since domain of f'= domain of f, we will only show that $f'(C) = f(C)$ for all $C \in \mathcal{C}$. Let $C \in \mathcal{C}$. Since g is surjective, there exists $A \in \mathcal{A}$ such that $g(A) = C$. Then $f'(C) = f'(g(A)) = f'g(A) = h(A) = fg(A) = f(g(A)) = f(C)$.
(iii)($\Rightarrow$) Let f in (1) is injective and $(A, B) \in kerh$. Then $h(A) = h(B)$. Hence $fg(A) = fg(B)$. So $f(g(A)) = f(g(B))$. Then $g(A) = g(B)$. That is $(A, B) \in kerg$. So $kerh \subseteq kerg$. Then $kerg = kerh$.
($\Leftarrow$) Let $kerg = kerh$ and $C_1, C_2 \in \mathcal{C}$ such that $f(C_1) = f(C_2)$. Since g is surjective, there exists $A_1, A_2 \in \mathcal{A}$ such that $g(A_1) = C_1$ and $g(A_2) = C_2$. Then $h(A_1) = fg(A_1) = f(g(A_1)) = f(C_1) = f(C_2) = f(g(A_2)) = fg(A_2) = h(A_2)$. That is $(A_1, A_2) \in kerh = kerg$. Then $g(A_1) = g(A_2)$. So $C_1 = C_2$.
(iv)($\Rightarrow$) Let f in (1) is surjective and $B \in \mathcal{B}$. There exists $C \in \mathcal{C}$ such that $f(C) = B$. Since g is surjective, there exists $A \in \mathcal{A}$ such that $g(A) = C$. Consider $h(A) = fg(A) = f(g(A)) = f(C) = B$. Then for any $B \in \mathcal{B}$, there is $A \in \mathcal{A}$ such that $h(A) = B$.
($\Leftarrow$) Let h is surjective and $B \in \mathcal{B}$. There exists $A \in \mathcal{A}$ such that $h(A) = B$. Then $g(A) = C$ for some $C \in \mathcal{C}$. So $f(C) = f(g(A)) = fg(A) = h(A) = B$. Then for any $B \in \mathcal{B}$. There exists $C \in \mathcal{C}$ such that $f(C) = B$.

Theorem 3.17 *Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Then there exists a unique isomorphism $f : \mathcal{A}/kerh \to \mathcal{B}$ such that $fnat_{kerh} = h$.*

Proof Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism from $\mathcal{A}$ into $\mathcal{B}$. By Theorem 3.15, we have nat_{kerh} is a surjective homomorphism from $\mathcal{A}$ into $\mathcal{A}/kerh$ define by $nat_{kerh}(A) = A/kerh$ for all $A \in \mathcal{A}$. We will show that $ker(nat_{kerh}) = kerh$. Consider $(A_1, A_2) \in ker(nat_{kerh}) \Leftrightarrow nat_{kerh}(A_1) = nat_{kerh}(A_2) \Leftrightarrow A_1/kerh = A_2/kerh \Leftrightarrow h(A_1) = h(A_2) \Leftrightarrow (A_1, A_2) \in kerh \Leftrightarrow ker(nat_{kerh}) \subseteq kerh$. Then $ker(nat_{kerh}) = kerh$. Since $ker(nat_{kerh}) = kerh$ and h is surjective, by Theorem 3.16, there exists a unique isomorphism $f : \mathcal{A}/kerh \to \mathcal{B}$ such that $fnat_{kerh} = h$.

Theorem 3.18 *Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Let $\mathcal{A}_0$ be a subsemigroup of $\mathcal{A}$ and $\mathcal{A}_0^* = h^{-1}(h(\mathcal{A}_0))$. Let $h_0 = h \mid_{\mathcal{A}_0}$ and $h_0^* = h \mid_{\mathcal{A}_0^*}$ be restriction of h on $\mathcal{A}_0$ and $\mathcal{A}_0^*$ respectively. Then $\varphi : \mathcal{A}_0/kerh_0 \to \mathcal{A}_0^*/kerh_0^*$ define by $\varphi(A/kerh_0) = A/kerh_0^*$ is isomorphism.*

Proof Let $(\mathcal{A}, \cdot)$, $(\mathcal{B}, \cdot')$ be power semigroups on semihypergroups $(H, \circ)$ and $(H', \circ')$ respectively. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism and $\mathcal{A}_0$ be a subsemigroup of $\mathcal{A}$. By Theorem 3.11(i), $h(\mathcal{A}_0) \subseteq \mathcal{B}$. Let $\mathcal{B}_0 = h(\mathcal{A}_0)$. Let $\mathcal{A}_0^* = h^{-1}(h(\mathcal{A}_0))$, $h_0 = h \mid_{\mathcal{A}_0}$ and $h_0^* = h \mid_{\mathcal{A}_0^*}$. Then $\mathcal{A}_0^* = h^{-1}(h(\mathcal{A}_0)) = h^{-1}(\mathcal{B}_0)$. By Theorem 3.11(ii), $\mathcal{A}_0^*$ is a subsemigroup of $\mathcal{A}$. Since $\mathcal{A}_0 \subseteq h^{-1}(h(\mathcal{A}_0)) = \mathcal{A}_0^* \subseteq \mathcal{A}$, $\mathcal{A}_0$ be a subsemigroup of $\mathcal{A}$ and $\mathcal{A}_0^*$ is a subsemigroup of $\mathcal{A}$ then $\mathcal{A}_0$ is a subsemigroup of $\mathcal{A}_0^*$. Since $h_0 : \mathcal{A}_0 \to \mathcal{B}_0$ and $h_0^* : \mathcal{A}_0^* \to \mathcal{B}_0$ be a surjective homomorphism, by Theorem 3.15, there exists a unique isomorphism $f : \mathcal{A}_0/kerh_0 \to \mathcal{B}_0$ and $g : \mathcal{A}_0^*/kerh_0^* \to \mathcal{B}_0$ such that $fnat_{kerh_0} = h_0$ and $gnat_{kerh_0^*} = h_0^*$. We will show that $\varphi : \mathcal{A}_0/kerh_0 \to \mathcal{A}_0^*/kerh_0^*$ define by $\varphi(A/kerh_0) = A/kerh_0^*$ is a homomorphism. (well-define) Let $A/kerh_0, B/kerh_0 \in \mathcal{A}_0/kerh_0$ such that $A/kerh_0 = B/kerh_0$. Hence $(A, B) \in kerh_0$. Then $h_0(A) = h_0(B)$. Since $h_0(A), h_0(B) \in \mathcal{B}_0$, $A, B \in \mathcal{A}_0 \subseteq \mathcal{A}_0^*$ and $h_0(A) = h_0^*(A)$, $h_0(B) = h_0^*(A)$. Hence $h_0^*(A) = h_0^*(A)$. Then $(A, B) \in kerh_0^*$. So $A/kerh_0^* = B/kerh_0^*$. Therefore φ is well-define. We will show that φ is isomorphism by show that $\varphi = g^{-1}f$. Let $A/kerh_0 \in \mathcal{A}_0/kerh_0$. Consider $(g^{-1}f)(A/kerh_0) = g^{-1}(f(A/kerh_0)) = g^{-1}(f(nat_{kerh_0}(A))) = g^{-1}(fnat_{kerh_0}(A)) = g^{-1}(h_0(A))$. Since $h_0(A) \in \mathcal{B}_0$, $A \in \mathcal{A}_0^*$ and $h_0^*(A) = h_0^*(A)$. So $(g^{-1}f)(A/kerh_0) = g^{-1}(h_0(A)) = g^{-1}(h_0^*(A)) = g^{-1}(gnat_{kerh_0^*}(A)) = g^{-1}g(nat_{kerh_0^*}(A)) = nat_{kerh_0^*}(A) = A/kerh_0^* = \varphi(A/kerh_0)$. Then $\varphi = g^{-1}f$. Since g is isomorphism, g^{-1} is isomorphism. Since f, g^{-1} is isomorphism, $g^{-1}f$ is isomorphism. Therefore $\varphi : \mathcal{A}_0/kerh_0 \to \mathcal{A}_0^*/kerh_0^*$ is isomorphism.

Definition 3.19 *Let* $(\mathcal{A}, \cdot)$ *be a power semigroup on a semihypergroup* $(H, \circ)$ *and* θ_1, θ_2 *be congruence on* $\mathcal{A}$ *such that* $\theta_1 \subseteq \theta_2$. *Define a relation on* $\mathcal{A}/\theta_1$ *by* $\theta_2/\theta_1 = \{(A/\theta_1, B/\theta_1) \mid (A, B) \in \theta_2\}$.

Theorem 3.20 *Let* $(\mathcal{A}, \cdot)$ *be a power semigroup on a semihypergroup* $(H, \circ)$ *and* θ_1, θ_2 *be congruence on* $\mathcal{A}$ *such that* $\theta_1 \subseteq \theta_2$. *Then* θ_2/θ_1, *defined as in Theorem 3.19 is a congruence relation on* $(\mathcal{A}/\theta_1, \star_1)$ *and* $\varphi : (\mathcal{A}/\theta_1)/(\theta_2/\theta_1) \to \mathcal{A}/\theta_2$, *defined by* $\varphi((A/\theta_1)/(\theta_2/\theta_1)) = A/\theta_2$, *is isomorphism.*

Proof Let $(\mathcal{A}, \cdot)$ be a power semigroup on a semihypergroup $(H, \circ)$ and θ_1, θ_2 be congruence on $\mathcal{A}$ such that $\theta_1 \subseteq \theta_2$. Define a relation on $\mathcal{A}/\theta_1$ by $\theta_2/\theta_1 = \{(A/\theta_1, B/\theta_1) \mid (A, B) \in \theta_2\}$. By the definition of θ_2/θ_1, θ_2/θ_1 is an equivalence relation on $\mathcal{A}/\theta_1$. We will show that $\star_1$ is compatible with θ_2/θ_1. Let $(A_1/\theta_1, B_1/\theta_1), (A_2/\theta_1, B_2/\theta_1) \in \theta_2/\theta_1$. Then $(A_1, B_1), (A_2, B_2) \in \theta_2$. Since θ_2 is a congruence on $\mathcal{A}$, $(A_1 \cdot A_2, B_1 \cdot B_2) \in \theta_2$. Hence $((A_1 \cdot A_2)/\theta_1, (B_1 \cdot B_2)/\theta_1) \in \theta_2/\theta_1$. So $(A_1/\theta_1 \star_1 A_2/\theta_1, B_1/\theta_1 \star_1 B_2/\theta_1) \in \theta_2/\theta_1$. Then θ_2/θ_1 is a congruence on $\mathcal{A}/\theta_1$. Since θ_1, θ_2 are congruence on $\mathcal{A}$, $nat_{\theta_1} : \mathcal{A} \to \mathcal{A}/\theta_1$ and $nat_{\theta_2} : \mathcal{A} \to \mathcal{A}/\theta_2$ are surjective homomorphism. By Theorem 3.17, there exists a unique isomorphism $f : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ such that $fnat_{\theta_1} = nat_{\theta_2}$. Since θ_2/θ_1 is a congruence relation on $\mathcal{A}/\theta_1$, $nat_{\theta_2/\theta_1} : \mathcal{A}/\theta_1 \to (\mathcal{A}/\theta_1)/(\theta_2/\theta_1)$ is a surjective homomorphisms. Since $f : \mathcal{A}/\theta_1 \to \mathcal{A}/\theta_2$ is isomorphism, by Theorem 3.17, there exists a unique isomorphism $\psi : (\mathcal{A}/\theta_1)/(\theta_2/\theta_1) \to \mathcal{A}/\theta_2$ such that $\varphi nat_{\theta_2/\theta_1} = f$. That is, $\psi nat_{\theta_2/\theta_1} nat_{\theta_1} = fnat_{\theta_1} = nat_{\theta_2}$. Next, we will show that $\varphi : (\mathcal{A}/\theta_1)/(\theta_2/\theta_1) \to \mathcal{A}/\theta_2$, defined by $\varphi((A/\theta_1)/(\theta_2/\theta_1)) = A/\theta_2$, is well-defined. Let $(A/\theta_1)/(\theta_2/\theta_1), (B/\theta_1)/(\theta_2/\theta_1) \in (\mathcal{A}/\theta_1)/(\theta_2/\theta_1)$ such that $(A/\theta_1)/(\theta_2/\theta_1) = (B/\theta_1)/(\theta_2/\theta_1)$. Then $(A/\theta_1, B/\theta_1) \in \theta_2/\theta_1$. Hence $(A, B) \in \theta_2$ and $A/\theta_2 = B/\theta_2$. Therefore φ is well-defined. Next, we will show that φ is isomorphism by show that $\varphi = \psi$. Since domain of φ= domain of ψ, we will only show that $\varphi((A/\theta_1)/(\theta_2/\theta_1)) = \psi((A/\theta_1)/(\theta_2/\theta_1))$ for any $(A/\theta_1)/(\theta_2/\theta_1) \in (\mathcal{A}/\theta_1)/(\theta_2/\theta_1)$. Let $(A/\theta_1)/(\theta_2/\theta_1) \in (\mathcal{A}/\theta_1)/(\theta_2/\theta_1)$. Consider $\varphi((A/\theta_1)/(\theta_2/\theta_1)) = A/\theta_2 = nat_{\theta_2}(A) = (fnat_{\theta_1})(A) = ((\psi nat_{\theta_2/\theta_1})nat_{\theta_1})(A) = (\psi nat_{\theta_2/\theta_1})(nat_{\theta_1}(A)) = (\psi nat_{\theta_2/\theta_1})(A/\theta_1) = \psi(nat_{\theta_2/\theta_1}(A/\theta_1)) = \psi((A/\theta_1)/(\theta_2/\theta_1))$. Therefore φ is isomorphism.

4 Open Problems

In the presented paper, we concluded a new algebraic structure which is called a power semigroup on a semihypergroup and gave some of its algebraic properties such as homomorphism theorems and isomorphism theorems. In the future work, we can consider this structure together with a partial order relation on it, i.e. a power-ordered semigroup on a semihypergroup and study its algebraic properties.

ACKNOWLEDGEMENTS. This research was supported by Development and Promotion of Science and Technology Talents Project (DPST) and Chiang Mai University.

References

[1] N. G. Alimov, On ordered semigroups, *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 14(6) (1950) 569–576.

[2] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered sets, *Proceedings of the Summer School on General Algebra and Ordered Sets 1994* (1994) 19–30.

[3] A. H. Clifford, Totally ordered commutative semigroups, *Bulletin of the American Mathematical Society* 64(6) (1958) 305–316.

[4] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups, Volume I*, Mathematical Surveys and Monographs 7 (1961).

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani editore (1993).

[6] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Advanced in Mathematics, Springer (2003).

[7] B. Davvaz, *Semihypergroup Theory*, Academic Press (2016).

[8] B. Davvaz, P. Corsini, T. Changphas, Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder, *European Journal of Combinatorics* 44 (2015) 208–217.

[9] A. Hasankhani, Ideals in a semihypergroup and Greens relations, *Ratio Mathematica* 13 (1999) 29–36.

[10] D. Heidari, B. Davvaz, On ordered hyperstructures, *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics* 73(2) (2011) 85–96.

[11] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press (1995).

[12] J. Jantosciak, Transposition Hypergroups: Noncommutative Join Spaces, *Journal of algebra* 187(11) (1997) 97–119.

[13] N. Kehayopulu, On completely regular ordered semigroups, *Scinetiae Mathematicae* 1(1) (1998) 27–32.

[14] N. Kehayopulu, On completely regular *poe*-semigroups, *Mathematica Japonica* 37(1) (1992) 123–130.

[15] N. Kehayopulu, M. Tsingelis, A note on ordered groupoids-semigroups, *Scientiae Mathematicae* 3(2) (2000) 251–255.

[16] N. Kehayopulu, M. Tsingelis, On intra-regular ordered semigroups, *Semigroup Forum* 57(1) (1998) 138–141.

[17] N. Kehayopulu, M. Tsingelis, On subdirectly irreducible ordered semigroups, *Semigroup Forum* 50(2) (1995) 161–177.

[18] N. Kehayopulu, M. Tsingelis, On weakly commutative ordered semigroups, *Semigroup Forum* 56(1) (1998) 32–35.

[19] N. Kehayopulu, G. Lepouras, M. Tsingelis, On right regular and right duo ordered semigroups, *Mathematica japonicae* 46(2) (1997) 311–315.

[20] H. Li, P. Wang, Power group, *Applied Mathematics* 1(1) (1988) 1–4.

[21] S. Ma, H. Mi, L. Huo, Power Group on Hypergroup, *Proceedings of 2010 Seventh international conference on fuzzy systems and knowledge discovery(FSKD2010), IEEE* (2010) 269–274.

[22] F. Marty, Sur Une Generalization de Group, *Proceedings of the 8th Congres Des Mathematiciens Scandinaves, Stockholm, Sweden, 14-18 August 1934* (1934) 45–49.

[23] T. Saito, Ordered idempotent semigroups, *Journal of the Mathematical Society of Japan* 14(2) (1962) 150–169.

[24] T. Saito, Regular elements in an ordered semigroup, *Pacific Journal of Mathematics* 13 (1963) 263–295.

[25] M. Szymanska, D. Schweigert, Structures on the Power set, *Proceedings of the sixth international conference on discrete mathematics and applications, 31.08.2001-02.09.2001, Bansko, Bulgaria* (2001) 71–76.

[26] B. Yao, The Structures of Power Groups, *Journal of the Tripura Mathematical Society* 3 (2001) 1–6.