

# Quantum Partial Derivatives of $Q$ -analytic Functions On Quantum Superspace $A_q$

Marzieh Mostafavi

Department of Mathematics, University of Qom, Qom, Iran  
e-mail:mmostafavi14279@gmail.com

Received 22 February 2022; Accepted 28 March 2022

## Abstract

*A class of  $Q$ -analytic functions on quantum plane was introduced by Abad and Milani in 1994. As a multiparametric deformation of it, in this manuscript, we construct the  $Q$ -analytic functions on the quantum superspace described by Manin in 1989. Then we define the quantum partial derivatives  $\partial_q/\partial x_i$  of the  $Q$ -analytic functions of  $O_q$ , for  $i = 1, \dots, n$  and any arbitrary format and any multiparameter  $q$ .*

**Keywords:** *Quantum superspace;  $Q$ -analytic functions; Quantum partial derivatives.*

**2010 Mathematics Subject Classification:** 05A30, 46L165, 81T13, 81T75.

## 1 Introduction

Physics describes nature in two supplement modes: classical and quantum. The notion of using the possibilities of quantum mechanics in organizing computation, has been attractive for years. The stimulus to peruse quantum superspaces comes from several sources: physics and technology, cognitive science, and mathematics. It should be said as a physical history that the effects of two-photon interference when incidental photons are in the same polar state can only be explained by quantum mechanics [9]. Two-level system can be presented as a quantum bit or "qubit" and the interaction between these systems can lead to the creation of quantum gates that follow non-classical logic [8]. Superposition in multi-particle systems creates phenomena that are far more interesting than anything seen in a particle system. One can understand the

ramifications of the superposition principle at the heart of quantum mechanics (see [11]). Quantum theory of optical coherence is described and discussed three major applications of general relativity: black holes, perturbation theory and gravitational waves, and cosmology [10]. The mathematical model must refer only to the general principles of quantum physics, without prescribing a physical execution [16].

We recommend as two general references on quantum information theory, the books [12, 13] and as the first acquaintance with the mathematical theory of quantum computation, the book of Kitaev [16]. (see also [3])

Quantum groups, quantum cohomology, quantum invariants of knots, are three regions of quantization in mathematics. Woronowicz [22] precisely defined the compact matrix pseudogroup which is a non-commutative compact space dedicated with a group structure. In the meantime he had compact group of matrices, duals of discrete groups and twisted (deformed)  $SU(N)$  groups. Integrable models in (1+1)-dimensional quantum field theory were mentioned by Faddeev [6] and then he presented Hamiltonian methods in the theory of solitons [7]. In the works of Drinfeld [4, 5], Faddeev [20] quantum groups were introduced and a new class of Hopf algebras was constructed. For Manin, the starting point of the theory of quantum groups was the quadratic algebras. He has described the general quantum groups in [17, 18] as the symmetries of non-commutative algebraic varieties defined by quadratic equations:

Let  $q$  be an invertible real (complex) number less than 1, and let  $I_q$  be the two-sided ideal of the free algebra  $k[x, y]$  generated by the element  $yx - qxy$ . We can think of  $q$  as a deformation parameter measuring the coupling among different spatial degrees of freedom. The quantum plane is define as the quotient-algebra

$$k_q[x, y] = k\{x, y\}/I_q. \quad (1)$$

As a multiparametric deformation of these quantum groups, Manin [19] has introduced the universal Hopf (super)algebra  $H_q$  generated by the entires of a multiplicative matrix  $E_q$  and two dual quantum superspaces  $A_q$  and  $A_q^*$  as comodule-algebras over it:

Let  $k$  be a field of characteristic  $\neq 2$  ( $\mathbb{R}$  or  $\mathbb{C}$ ). Suppose the format  $(a_1, \dots, a_n)$  be an arbitrary sequence,  $a_i \in \mathbb{Z}_2$  and the multiparameter

$$q = \{q_{ij} : |q_{ij}| \leq 1, \quad 1 \leq i, j \leq n\}, \quad (2)$$

be the family of non-zero elements of  $k$ . The quantum superspace  $A_q$  or, rather, the polynomial function ring on it is generated by coordinates  $x_1, \dots, x_n$  with parity  $\hat{x}_i = \hat{i}$  and commutation rules:

$$x_i^2 = 0 \quad \text{for } \hat{i} = 1 \quad \text{and} \quad x_j x_i = (-1)^{\hat{i}\hat{j}} q_{ij} x_i x_j \quad \text{for } i < j. \quad (3)$$

Abad and Milani [1] have defined  $Q$ -analytic functions on the quantum plane as a quantization of the  $\mathbb{R}$ -algebra of the absolutely convergent power

series

$$\sum_{i,j >> -\infty} a_{ij} t_1^i t_2^j \quad \text{on} \quad \mathbb{R} - \{0\} \times \mathbb{R} - \{0\}. \tag{4}$$

Since  $q$ -series have extensive applications and appear in combinatorics, in number theory, in statistical mechanics, and in theory of Lie algebras, we define in this manuscript, the  $Q$ -analytic functions on the quantum superspace  $A_q$  as a multiparametric deformation of (??). Then we introduce the  $Q$ -algebras with format  $(a_1, \dots, a_n)$  on any  $k$ -vector space and on any  $B$ -module where  $B$  is a  $k$ -algebra. The  $n$ th quantum derivative  $\mathcal{D}_n f(x)$  of the real-valued function  $f$  at each non-zero real number  $x$ , was differently defined in many letters such as [2, 21, 13].

In this paper, we introduce the quantum partial derivatives of the  $Q$ -analytic functions on the multiparametric deformed quantum superspace  $A_q$ , independently of [1] and [2] but as a multiparametric deformation of the definition in Kassel's book [15]

## 2 Preliminaries

For the benefit of the reader we summarize some of the relevant background material from [15]

Let  $\alpha$  is the automorphism of the polynomial ring  $k[x]$  determined by  $\alpha(x) = qx$  where  $|q| < 1$ , then the algebra  $k_q[x, y]$  is isomorphic to the Ore extension  $k[x][y, \alpha, \sigma]$  and using  $yx = qxy$ , we have for any pair  $(m, k)$  of nonnegative integers:

$$y^k x^m = q^{mk} x^m y^k. \tag{5}$$

Given any  $k$ -algebra  $R$ , there is a natural bijection

$$Hom_{A_q}(k_q[x, y], R) \cong \{(X, Y) \in R \times R \mid YX = qXY\}.$$

A pair  $(X, Y)$  of elements of  $R$  subject to the relation  $YX = qXY$  will be called an  $R$ -point of the quantum plane. Let  $A$  be the algebra of complex function on  $\mathbb{C} \setminus \{0\}$  and let  $q \neq 0, 1$  be a complex number. Consider the linear endomorphisms  $\tau_q$  and  $\delta_q$  in  $R = End(A)$  defined by:

$$\tau_q(f)(x) = f(qx) \quad \text{and} \quad \delta_q(f)(x) = \frac{f(qx) - f(x)}{(qx - x)}.$$

The pair  $(\tau_q, \delta_q)$  is an  $R$ -point of  $k_q[x, y]$ . The limit of operator  $\delta_q$  when  $q$  tends to 1 is the usual derivative  $d/dx$ .

Given two automorphisms  $\sigma$  and  $\tau$  of an algebra  $A$ , a linear endomorphism  $\delta$  of  $A$  is called a  $(\sigma, \tau)$ -derivation if

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)\tau(b), \tag{6}$$

for all  $a, b$  in  $A$ . The  $q$ -analogues  $\partial_q/\partial x$  and  $\partial_q/\partial y$  on quantum space in the case  $n = 2$  and with format  $(0, 0)$  is defined as:

$$\frac{\partial_q}{\partial x}(x^m y^k) = [m] x^{m-1} y^k \quad \text{and} \quad \frac{\partial_q}{\partial y}(x^m y^k) = [k] x^m y^{k-1}. \quad (7)$$

$$\text{Where } [m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

### 3 Main results

$Q$ -algebra  $O_q$  with format  $(a_1, \dots, a_n)$

**Definition 3.1** *The algebra of the germs of  $Q$ -analytic functions on quantum superspace  $A_q$ , denoted by  $O_q$ , is obtained by replacing the ordinary coordinates with  $q$ -coordinates with parity  $(a_1, \dots, a_n)$  and commutation relations (3) in the Taylor expansion of any analytic function around any point of  $\mathbb{R}^n$  on its interval convergence.*

For simplicity we denote around  $(0, \dots, 0)$ :

$$f(t_1, \dots, t_n) \longrightarrow \hat{f}(x_1, \dots, x_n);$$

$$\sum_{k_1, \dots, k_n \geq 0} \alpha_{k_1, \dots, k_n} t_1^{k_1} \dots t_n^{k_n} \longrightarrow \sum_{k_1, \dots, k_n \geq 0} \alpha_{k_1, \dots, k_n} x_1^{k_1} \dots x_n^{k_n}. \quad (8)$$

We call  $x_i$  an odd (even) coordinate  $\hat{i} = 1$  ( $\hat{i} = 0$ ) and we have  $k_i = 0, 1$  ( $k_i \geq 0$ ). We denote briefly an  $Q$ -analytic function by  $\hat{f}(x) = \sum_{k \geq 0} \alpha_k x^k$ .

**Proposition 3.2** *The multiplication of any two  $Q$ -analytic functions using (3), can be briefly written:*

$$\hat{f}(x) \cdot \hat{g}(x) = \sum_{k \geq 0} \alpha_k x^k \cdot \sum_{l \geq 0} \beta_l x^l = \sum_{k, l \geq 0} \alpha_k \beta_l \gamma_{kl} x^{k+l}. \quad (9)$$

where

$$\gamma_{kl} = \prod_{1 \leq i \leq n-1} \prod_{i < j \leq n} (-1)^{l_i k_j \hat{i} \hat{j}} q_{ij}^{l_i k_j}.$$

**Proof**

$$\begin{aligned} x_1^{k_1} \dots x_n^{k_n} x_1^{l_1} \dots x_n^{l_n} &= (x_1^{k_1} \dots x_n^{k_n}) (x_1^{l_1} \dots x_n^{l_n}) \\ &= \left( \prod_{1 < j \leq n} (-1)^{l_1 k_j \hat{1} \hat{j}} q_{1j}^{l_1 k_j} \right) (x_1^{k_1+l_1} x_2^{k_2} \dots x_n^{k_n}) (x_2^{l_2} \dots x_n^{l_n}) \\ &= \left( \prod_{1 \leq i \leq n-1} (-1)^{l_i k_j \hat{1} \hat{j}} \prod_{i < j \leq n} q_{ij}^{l_i k_j} \right) (x_1^{k_1+l_1} \dots x_n^{k_n+l_n}) \\ &= \gamma_{kl} x_1^{(k_1+l_1)} \dots x_n^{(k_n+l_n)}. \end{aligned}$$

Since  $f(t).g(t) = \sum_{k,l \geq 0} \alpha_k \beta_l t^{k+l}$  is absolutely convergent, then the power series  $\sum_{k,l \geq 0} \alpha_k \beta_l \gamma_{kl} t^{k+l}$  is absolutely convergent. Therefore  $\hat{f}(x).\hat{g}(x)$  is an  $Q$ -analytic function.

**Definition 3.3** An automorphism of the  $Q$ -algebra  $O_q$  is an automorphism of  $k$ -algebras

$$\left\{ \begin{array}{l} \psi : O_q \longrightarrow O_q \\ \sum \alpha_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n} \longrightarrow \sum \alpha_{k_1 \dots k_n} (\psi(x_1))^{k_1} \dots (\psi(x_n))^{k_n}. \end{array} \right.$$

Such that  $\psi(x_1), \dots, \psi(x_n)$  are coordinates on quantum superspace satisfying the rules (3). There is a one to one correspondence between automorphisms of  $O_q$  and automorphisms of  $A_q$ .

**$Q$ -algebras with format  $(a_1, \dots, a_n)$  on  $k$ -vector spaces and  $B$ -modules**

**Definition 3.4** Let  $V$  be a  $k$ -vector space of dimension  $n$  with the basis  $\{v_1, \dots, v_n\}$ . The  $Q$ -algebra with format  $(a_1, \dots, a_n)$  on  $V$ , denoted by  $O_q V$  is the set of all formal series

$$\sum_{k_1, \dots, k_n \geq 0} \alpha_{k_1, \dots, k_n} v_1^{k_1} \dots v_n^{k_n}$$

where  $\sum_{k_1, \dots, k_n \geq 0} \alpha_{k_1, \dots, k_n} t_1^{k_1} \dots t_n^{k_n}$  is an absolutely convergent series around  $(0, \dots, 0) \in k^n$ . In this algebra the commutation rules are

$$\begin{aligned} v_i^2 &= 0 \quad \text{for } \hat{i} = 1; \\ v_j v_i &= (-1)^{\hat{i}\hat{j}} q_{ij} v_i v_j \quad \text{for } i < j. \end{aligned}$$

Also the product of two elements of  $O_qV$  is similar to (9)

$$\sum_{k \geq 0} \alpha_k v^k \cdot \sum_{l \geq 0} \beta_l v^l = \sum_{k, l \geq 0} \alpha_k \beta_l \gamma_{kl} v^{k+l}.$$

If one puts  $q_{ij} = 1$  for all  $i, j$  and  $a_i = 1$  for all  $i$ , then  $O_qV = \Lambda V$ , the exterior algebra over  $V$ .

**Definition 3.5** Let  $W$  be an  $B$ -module with the basis  $(w_1, \dots, w_n)$  and  $B$  be a  $k$ -algebra. We define the  $Q$ -algebra with format  $(a_1, \dots, a_n)$  on  $W$  over  $B$  by  $(O_qW)_B = B \otimes O_qW$ . Each element of  $O_qW$  is in the form  $w = b_i \otimes f^i(w_1, \dots, w_n)$  where

$$f^i(w_1, \dots, w_n) = \sum_{k_1, \dots, k_n \geq 0} \alpha_{k_1 \dots k_n}^i w_1^{k_1} \dots w_n^{k_n}, \quad \alpha_{k_1 \dots k_n}^i \in k.$$

Notice when we write an index of a letter above the next letter in a product, we mean a finite summand of the product.

### Quantum derivatives of $Q$ -analytic functions on $A_q$

We wish to extend partial quantum derivations illustrated in formula 7 to the  $q$ -series by the following proposition:

**Proposition 3.6** The quantum partial derivatives  $\partial_q/\partial x$  and  $\partial_q/\partial y$  of any  $Q$ -analytic function  $\hat{f}(x, y) = \sum \alpha_{mk} x^m y^k$  on 2 dimensional quantum plane with the format  $(0, 0)$  can be defined Using (7), by:

$$\begin{aligned} \frac{\partial_q}{\partial x} \hat{f}(x, y) &= \sum \alpha_{mk} [m] x^{m-1} y^k, \\ \frac{\partial_q}{\partial y} \hat{f}(x, y) &= \sum \alpha_{mk} [m] x^m y^{k-1}, \end{aligned}$$

where  $yx = qxy$  and  $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ .

**Proof** Let  $f(t_1, t_2) = \sum \alpha_{mk} t_1^m t_2^k$  be an absolutely convergent series on

$$I = \{t_1 \in \mathbb{C} : |t_1| < r_1\} \times \{t_2 \in \mathbb{C} : |t_2| < r_2\}.$$

Then by ratio test, we have  $\rho = \lim_{m \rightarrow \infty} \left| \frac{\alpha_{mk}}{\alpha_{(m-1)k}} \right| < 1$  and since

$$\lim_{m \rightarrow \infty} \left| \frac{[m]}{[m-1]} \right| = \lim_{m \rightarrow \infty} \left| \frac{q^m - q^{-m}}{q^{m-1} - q^{-m+1}} \right| = \lim_{m \rightarrow \infty} \left| q + \frac{q^2 - 1}{q^{2m-1} - q} \right| = |q|,$$

and  $|q| < 1$ , hence  $\sum \alpha_{mk} [m] t_1^{m-1} t_2^k$  is absolutely convergent on  $I$ . Therefore  $\frac{\partial_q}{\partial x} \hat{f}(x, y)$  is a  $Q$ -analytic function.

**Theorem 3.7** Consider the  $Q$ -algebra  $O_q$  with format  $(0, \dots, 0)$ . We define quantum partial derivative  $\partial_q/\partial x_i$  for all  $1 \leq i \leq n$  by

$$\frac{\partial_q}{\partial x_i} (x_1^{m_1} \dots x_n^{m_n}) = [m_i] x_1^{m_1} \dots x_i^{m_i-1} \dots x_n^{m_n}, \quad (10)$$

Where

$$[m_i] = (q_i^{m_i} - q_i^{-m_i})/(q_i - q_i^{-1}) \quad \text{and} \quad q_i = \prod_{1 \leq j \leq n, j \neq i} q_{ij}. \quad (11)$$

If we define the  $Q$ -algebra automorphisms  $\sigma_i, \sigma'_i$  and  $\sigma''_i$  by

$$\sigma_i (x_1^{m_1} \dots x_n^{m_n}) = q_i^{m_i} x_1^{m_1} \dots x_n^{m_n}, \quad (12)$$

$$\sigma'_i (x_1^{m_1} \dots x_n^{m_n}) = \left( \prod_{1 \leq j < i} (-1)^{m_j \hat{1} \hat{j}} q_{ji}^{m_j} \right) x_1^{m_1} \dots x_n^{m_n}, \quad (13)$$

$$\sigma''_i (x_1^{m_1} \dots x_n^{m_n}) = \left( \prod_{i < j \leq n} (-1)^{m_j \hat{i} \hat{j}} q_{ij}^{m_j} \right) \dots x_n^{m_n}. \quad (14)$$

Then  $\partial_q/\partial x_1$  is a  $(\sigma_1^{-1} \sigma''_1, \sigma_1)$ -derivation and  $\partial_q/\partial x_i$  is a  $(\sigma_i \sigma''_i, \sigma'_i \sigma_i^{-1})$ -derivation of  $O_q$  for  $i \geq 2$ .

**Proof** Let

$$\begin{aligned} J &= \sigma_1^{-1} \sigma''_1 (x_1^{k_1} \dots x_n^{k_n}) \frac{\partial_q}{\partial x_1} (x_1^{l_1} \dots x_n^{l_n}) + \frac{\partial_q}{\partial x_1} (x_1^{l_1} \dots x_n^{l_n}) \sigma_1 (x_1^{l_1} \dots x_n^{l_n}) \\ &= q_1^{-k_1} \left( \prod_{1 < j \leq n} (-1)^{k_j \hat{1} \hat{j}} q_{1j}^{k_j} \right) (x_1^{k_1} \dots x_n^{k_n}) [l_1] (x_1^{\ell_1-1} x_2^{\ell_2} \dots x_n^{\ell_n}) \\ &\quad + [k_1] (x_1^{k_1-1} x_2^{k_2} \dots x_n^{k_n}) q_1^{\ell_1} (x_1^{l_1} \dots x_n^{l_n}) \end{aligned}$$

Since

$$\begin{aligned} &\left( \prod_{1 < j \leq n} (-1)^{k_j \hat{1} \hat{j}} q_{1j}^{k_j} \right) (x_1^{k_1} \dots x_n^{k_n}) (x_1^{\ell_1-1} x_2^{\ell_2} \dots x_n^{\ell_n}) \\ &= (x_1^{k_1-1} x_2^{k_2} \dots x_n^{k_n}) (x_1^{l_1} \dots x_n^{l_n}) \\ &= \left( \prod_{1 < j \leq n} (-1)^{l_j k_j \hat{1} \hat{j}} q_{1j}^{l_j k_j} \right) (x_1^{k_1+l_1-1} x_2^{k_2} \dots x_n^{k_n}) (x_2^{\ell_2} \dots x_n^{\ell_n}) \\ &= \gamma_{k\ell} (x_1^{k_1+l_1-1} x_2^{k_2+\ell_2} \dots x_n^{k_n+\ell_n}). \end{aligned}$$

Hence

$$J = \gamma_{kl} ( q_1^{-k_1} [\ell_1] + q_1^{l_1} [k_1] ) ( x_1^{k_1+l_1-1} x_2^{k_2+l_2} \dots x_n^{k_n+l_n} ).$$

On the other hand

$$\begin{aligned} q_1^{-k_1} [\ell_1] + q_1^{l_1} [k_1] &= q_1^{-k_1} \frac{q_1^{l_1} - q_1^{-l_1}}{q_1 - q_1^{-1}} + q_1^{l_1} \frac{q_1^{k_1} - q_1^{-k_1}}{q_1 - q_1^{-1}} \\ &= \frac{q_1^{k_1+l_1} - q_1^{-(k_1+l_1)}}{q_1 - q_1^{-1}} \\ &= [k_1 + l_1]. \end{aligned}$$

Therefore

$$\begin{aligned} J &= \gamma_{kl} [k_1 + l_1] ( x_1^{k_1+l_1-1} x_2^{k_2+l_2} \dots x_n^{k_n+l_n} ) \\ &= \gamma_{kl} \frac{\partial_q}{\partial x_1} ( x_1^{(k_1+l_1)} \dots x_n^{(k_n+l_n)} ) \\ &= \frac{\partial_q}{\partial x_1} ( x_1^{k_1} \dots x_n^{k_n} x_1^{l_1} \dots x_n^{l_n} ). \end{aligned}$$

We prove that  $\partial_q/\partial x_2$  is a  $(\sigma_2 \sigma_2'', \sigma_2' \sigma_2^{-1})$  – derivation and the proofs of  $\partial_q/\partial x_i$ ,  $i \geq 3$  are similar.

$$\begin{aligned} K &= \sigma_2 \sigma_2'' ( x_1^{k_1} \dots x_n^{k_n} ) \frac{\partial_q}{\partial x_2} ( x_1^{l_1} \dots x_n^{l_n} ) + \frac{\partial_q}{\partial x_2} ( x_1^{l_1} \dots x_n^{l_n} ) \sigma_2^{-1} \sigma_2' ( x_1^{l_1} \dots x_n^{l_n} ) \\ &= q_2^{k_2} \left( \prod_{2 < j \leq n} (-1)^{k_j \hat{2} j} q_{2j}^{k_j} \right) ( x_1^{k_1} \dots x_n^{k_n} ) [l_2] ( x_1^{l_1} x_2^{l_2-1} x_3^{l_3} \dots x_n^{l_n} ) \\ &\quad + [k_2] ( x_1^{k_1} x_2^{k_2-1} x_3^{k_3} \dots x_n^{k_n} ) q_2^{-l_2} \left( (-1)^{l_1 \hat{1} \hat{2}} q_{12}^{l_1} \right) ( x_1^{l_1} \dots x_n^{l_n} ). \end{aligned}$$

Since

$$\begin{aligned} &\left( \prod_{2 < j \leq n} (-1)^{k_j \hat{2} j} q_{2j}^{k_j} \right) ( x_1^{k_1} \dots x_n^{k_n} ) ( x_1^{l_1} x_2^{l_2-1} x_3^{l_3} \dots x_n^{l_n} ) \\ &= \gamma_{kl} x_1^{k_1+l_1} x_2^{k_2+l_2-1} x_3^{k_3+l_3} \dots x_n^{k_n+l_n} \\ &= ( x_1^{k_1} x_2^{k_2-1} x_3^{k_3} \dots x_n^{k_n} ) \left( (-1)^{l_1 \hat{1} \hat{2}} q_{12}^{l_1} \right) ( x_1^{l_1} \dots x_n^{l_n} ) \end{aligned}$$

then

$$\begin{aligned}
 K &= \left( q_2^{k_2} [l_2] + q_2^{-l_2} [k_2] \right) \gamma_{kl} x_1^{k_1+l_1} x_2^{k_2+l_2-1} x_3^{k_3+l_3} \dots x_n^{k_n+l_n} \\
 &= [l_2 + k_2] \gamma_{kl} x_1^{k_1+l_1} x_2^{k_2+l_2-1} x_3^{k_3+l_3} \dots x_n^{k_n+l_n} \\
 &= \frac{\partial_q}{\partial x_2} \left( \gamma_{kl} x_1^{k_1+l_1} \dots x_n^{k_n+l_n} \right) \\
 &= \frac{\partial_q}{\partial x_2} \left( x_1^{k_1} \dots x_n^{k_n} x_1^{l_1} \dots x_n^{l_n} \right).
 \end{aligned}$$

Furthermore, we can write down commutative counterparts of the relations.

**Theorem 3.8** *Using the notation above, we have the following commutation relations between  $\partial_q/\partial x_i$ ,  $\partial_q/\partial x_j$  and  $\sigma_i$ , for all  $1 \leq i, j \leq n$ :*

$$\frac{\partial_q}{\partial x_i} \sigma_i = q_i \sigma_i \frac{\partial_q}{\partial x_i} \quad \forall i, \tag{15}$$

$$\frac{\partial_q}{\partial x_j} \sigma_i = \sigma_i \frac{\partial_q}{\partial x_j} \quad i < j, \tag{16}$$

$$\frac{\partial_q}{\partial x_j} \frac{\partial_q}{\partial x_i} = \frac{\partial_q}{\partial x_i} \frac{\partial_q}{\partial x_j} \quad i < j. \tag{17}$$

**Proof** We perform the calculations to prove the relation (15)

$$\begin{aligned}
 \frac{\partial_q}{\partial x_i} \sigma_i(x_1^{m_1} \dots x_n^{m_n}) &= \frac{\partial_q}{\partial x_i} (q_i^{m_i} x_1^{m_1} \dots x_n^{m_n}) \\
 &= q_i^{m_i} [m_i] x_1^{m_1} \dots x_i^{m_i-1} \dots x_n^{m_n} \\
 &= q_i \sigma_i \left( \frac{\partial_q}{\partial x_i} x_1^{m_1} \dots x_n^{m_n} \right).
 \end{aligned}$$

and the proof of the relation (16) is similar. Also we prove the relation (17)

$$\begin{aligned}
\frac{\partial_q}{\partial x_j} \frac{\partial_q}{\partial x_i} (x_1^{m_1} \dots x_n^{m_n}) &= \frac{\partial_q}{\partial x_j} ([m_i] x_1^{m_1} \dots x_i^{m_i-1} \dots x_n^{m_n}) \\
&= [m_j][m_i] x_1^{m_1}, \dots, x_i^{m_i-1} \dots x_j^{m_j-1} \dots x_n^{m_n} \\
&= \frac{\partial_q}{\partial x_i} ([m_j] x_1^{m_1} \dots x_j^{m_j-1} \dots x_n^{m_n}) \\
&= \frac{\partial_q}{\partial x_i} \frac{\partial_q}{\partial x_j} (x_1^{m_1} \dots x_n^{m_n})
\end{aligned}$$

However, the action of partial quantum derivatives requires to modify the following commutation relations:

**Theorem 3.9** *Let  $x_{i\ell}$  ( $x_{ir}$ ) be the left (right) multiplication by the element  $x_i$  in  $O_q$ . The nontrivial commutation relations between the  $x_{i\ell}$ ,  $x_{ir}$ ,  $\sigma_i$ ,  $\sigma'_i$ ,  $\sigma''_i$  and  $\frac{\partial_q}{\partial x_i}$  can be described as the following:*

$$x_{j\ell} x_{i\ell} = (-1)^{\hat{i}\hat{j}} q_{ij} x_{i\ell} x_{j\ell} \quad i < j, \quad (18)$$

$$x_{ir} x_{jr} = (-1)^{\hat{i}\hat{j}} q_{ij} x_{jr} x_{ir} \quad i < j, \quad (19)$$

$$\sigma_i x_{i\ell} = q_i x_{i\ell} \sigma_i \quad \forall i, \quad (20)$$

$$\frac{\partial_q}{\partial x_1} x_{i\ell} = (-1)^{\hat{1}\hat{i}} q_{1i} x_{i\ell} \frac{\partial_q}{\partial x_1} \quad 1 < i, \quad (21)$$

$$\frac{\partial_q}{\partial x_1} x_{1\ell} = q_1^{-1} x_{1\ell} \frac{\partial_q}{\partial x_1} + \sigma_1, \quad (22)$$

$$\frac{\partial_q}{\partial x_i} x_{i\ell} = q_i x_{i\ell} \frac{\partial_q}{\partial x_i} + \sigma'_i \sigma_i^{-1} \quad i \geq 2, \quad (23)$$

$$\partial_q \partial x_1 x_{1r} = q_1 x_{1r} \frac{\partial_q}{\partial x_1} + \sigma_1^{-1} \sigma''_1, \quad (24)$$

$$\frac{\partial_q}{\partial x_i} x_{ir} = q_i^{-1} x_{ir} \frac{\partial_q}{\partial x_i} + \sigma_i \sigma''_i \quad i \geq 2, \quad (25)$$

$$x_{1\ell} \frac{\partial_q}{\partial x_1} = \frac{\sigma_1 - \sigma_1^{-1}}{q_1 - q_1^{-1}}, \quad (26)$$

$$x_{i\ell} \frac{\partial_q}{\partial x_i} = \frac{\sigma_i - \sigma_i^{-1}}{q_i - q_i^{-1}} \sigma'_i \quad i \geq 2. \quad (27)$$

**Proof** We calculate explicit formulas (21), (23), (27) for  $q$ -translations and others follow from similar computation.

$$\begin{aligned}
 \frac{\partial_q}{\partial x_1} x_{i\ell} (x_1^{k_1} \dots x_n^{k_n}) &= \frac{\partial_q}{\partial x_1} (x_i x_1^{k_1} \dots x_n^{k_n}) \\
 &= \sigma_1^{-1} \sigma_1''(x_i) \frac{\partial_q}{\partial x_1} (x_1^{k_1} \dots x_n^{k_n}) + \frac{\partial_q}{\partial x_1} (x_i) \sigma_1 (x_1^{k_1} \dots x_n^{k_n}) \\
 &= (-1)^{\hat{i}} q_{1i} (x_i) [k_1] (x_1^{k_1-1} x_2^{k_2} \dots x_n^{k_n}) \\
 &= (-1)^{\hat{i}} q_{1i} x_{i\ell} \frac{\partial_q}{\partial x_1} (x_1^{k_1} \dots x_n^{k_n})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial_q}{\partial x_i} x_{i\ell} (x_1^{k_1} \dots x_n^{k_n}) &= \frac{\partial_q}{\partial x_i} (x_i x_1^{k_1} \dots x_n^{k_n}) \\
 &= \sigma_i \sigma_i''(x_i) \frac{\partial_q}{\partial x_i} (x_1^{k_1} \dots x_n^{k_n}) + \frac{\partial_q}{\partial x_i} (x_i) \sigma_i' \sigma_i^{-1} (x_1^{k_1} \dots x_n^{k_n}) \\
 &= q_i x_i \frac{\partial_q}{\partial x_i} (x_1^{k_1} \dots x_n^{k_n}) + \sigma_i' \sigma_i^{-1} (x_1^{k_1} \dots x_n^{k_n}) \\
 &= \left( q_i x_{i\ell} \frac{\partial_q}{\partial x_i} + \sigma_i' \sigma_i^{-1} \right) (x_1^{k_1} \dots x_n^{k_n})
 \end{aligned}$$

$$\begin{aligned}
 x_{i\ell} \frac{\partial_q}{\partial x_i} (x_1^{k_1} \dots x_n^{k_n}) &= x_i [k_i] x_1^{k_1} \dots x_i^{k_i-1} \dots x_n^{k_n} \\
 &= [k_i] \left( \prod_{1 \leq j < i} q_{ij}^{k_j} \right) x_1^{k_1} \dots x_n^{k_n} \\
 &= [k_i] \sigma_i' (x_1^{k_1} \dots x_n^{k_n}) \\
 &= \frac{\sigma_i - \sigma_i^{-1}}{q_i - q_i^{-1}} \sigma_i' (x_1^{k_1} \dots x_n^{k_n})
 \end{aligned}$$

## 4 Open Problem

The open problem here is to examine under what conditions, we can introduce  $LG$ -fuzzy quantum partial derivatives of  $LG$ -fuzzy  $Q$ -analytic functions on  $LG$ -fuzzy quantum superspace  $A_q$  ?

## References

- [1] Abad, A. S. D., Milani, V. (1994),  $Q$ -analytic functions on quantum spaces, *J. Math. Phys.* **35**: 50-74.
- [2] Ash, J. M., Catoiu, Ríos-Collantess-de-Terán, S. R., (2002), On the  $n$ th quantum derivatives, *J. Lon. Math. Soci.* **66**(1): 114-130. sh, J. M., Catoiu, S., Ríos-Collantes-de-Terán, R. (2002). ON THE TH QUANTUM DERIVATIVE. *Journal of the London Mathematical Society*, 66(1), 114-130
- [3] K. Brown, K. R. Goodearl, *Lectures on algebraic quantum groups*, Birkhauser, 2002.
- [4] V.G. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation, *Sov. Math. Dokl.* **32** (1985) 254-258,
- [5] V.G. Drinfeld, Quantum groups, *Proc, Int. Congr. Math.*, Berkeley, **1** (1986), 798-820.
- [6] L.D. Faddeev, Integrable models in (1+1)-dimensional quantum field theory, (Lectures in Les Houches, 1982), Elsevier Science Publishers B:V., (1984).
- [7] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, (1987).
- [8] S. Haroche, J.-M. Raimond, Quantum Computing: Dream or Nightmare? *Physics Today*, **49**, 8, 51 (1996); doi.org/10.1063/1.881512
- [9] C. H. Holbrow, E. Galvez, M. E. Parks, Photon quantum mechanics and beam splitters, *Article in American Journal of Physics* (2002) DOI: 10.1119/1.1432972
- [10] R. J. Glauber, *Quantum Theory of Optical Coherence: Selected Papers and Lectures*, Wiley-VCH Verlag GmbH Co. KGaA, Weinheim, (2007) DOI:10.1002/9783527610075
- [11] D. M. Greenberger, M. A. Horne, A. Zeilinger, Multiparticle Interferometry and the Superposition Principle, *Physics Today* **46**, 8, 22 (1993);https://doi.org/10.1063/1.881360
- [12] J. Gruska, *Quantum Computing*, McGraw-Hill, London, (1999).
- [13] D. Gurevich, P. Saponov, Derivatives in noncommutative calculus and deformation property of quantum algebras, (2015). arXiv:1412.4014 [math.QA]

- [14] M. Jimbo, Quantum  $r$  matrix for the generalized Toda system, *Commun. Math. Phys.* **102** (1986) 537-547
- [15] Ch. Kassel, Quantum groups, Springer Verlag, 1995.
- [16] A. Kitaev, Classical and quantum computations, Lecture notes, Independent University, Moscow, 1998.
- [17] Yu.I. Manin, Some remarks on Koszul algebras and quantum groups, *Ann. Inst. Fourier, Tom XXXVII*, **4** (1987), 191-205.
- [18] Yu.I. Manin, Quantum groups and non-commutative geometry, Preprint Montreal University, CRM-1561, 1988.
- [19] Yu.I. Manin, Multiparametric quantum deformation of the general linear supergroup, *Commun. Math. Phys.* **123** (1989), 163-175.
- [20] N. Yu. Reshetikhin, L. A. Takhtadzhyan, L. D. Faddeev, Quantization of Lie groups and Lie algebras, *Algebra i Analiz*, **1(1)** (1989), 178–206; *Leningrad Math. J.*, **1(1)** (1990), 193–225.
- [21] H. Wachter, Analysis on  $q$ -deformed quantum spaces, *International Journal of Modern Physics*, **22(1)**, (2007), 95-164.
- [22] S.L. Woronowicz, Compact matrix pseudogroups, *Math. Phys.* **111** (1987), 613-665.