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Convolution Properties of Subclasses of Meromorphic Univalent Functions of Complex Order

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Abstract

In the paper, we introduce subclasses of meromorphic functions by using principle of subordination and find necessary and sufficient condition for coefficients of functions $\mathcal{F}$ to belonge to these classes.

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1 Introduction

Let Σ be the class of meromorphic functions

$$(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in $U^* = U/\{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$.

For $0 \leq \alpha < 1, \in \Sigma$ is called meromorphically starlike (convex) of order α iff

$$-Re\left\{\frac{z^{'}(z)}{(z)}\right\} > \alpha, \tag{2}$$

$$-Re\left\{1 + \frac{z^{''}(z)}{^{'}(z)}\right\} > \alpha, \tag{3}$$

and it is denoted by $\Sigma^*(\alpha)$ ($\Sigma^c(\alpha)$) [7] (see also [1, 2], [9, 10], [12]).

Tang et al. [11] defined the $q-$derivative $\partial_q((z))$ by:

$$\begin{aligned}\partial_q(z) &= \frac{(z)-(qz)}{(1-q)z}\\ &= -\frac{1}{qz^2}+\sum_{k=1}^{\infty}[k]_q a_k z^{k-1},\end{aligned}\tag{4}$$

where

$$[j]_q = \frac{1-q^j}{1-q}.\tag{5}$$

As $q \to 1^-$, $[j]_q = j$ and $\partial_q(z) =' (z)$. Also, we have

$$\begin{aligned}[k+1]_q &= [k]_q + q^k = q[k]_q + 1,\\ [k-1]_q &= q^{-1}[k]_q - q^{-1},\\ [0]_q &= [1]_q = 1,\end{aligned}$$

$$(z) = (z) * \frac{1}{z(1-z)}, \ -qz\partial_q(z) = (z) * \left[\frac{1-(q+1)z}{z\,(1-z)\,(1-qz)}\right].\tag{6}$$

Aouf et al. [4] defined the operator $N^n_{\lambda,q}(z)$, for $\in \Sigma$, $\lambda \geq 0$, $0 < q < 1$ by:

$$\mathcal{N}^0_{\lambda,q}(z) = (z),$$

$$\mathcal{N}^1_{\lambda,q}(z) = (1-\lambda)(z) + \frac{\lambda}{z}\partial_q(z^2(z)),$$

and for $n \in \mathbb{N} = \{1, 2, 3, ...\}$,

$$\begin{aligned}\mathcal{N}^n_{\lambda,q}(z) &= (1-\lambda)(\mathcal{N}^{n-1}_{\lambda,q}(z)) + \frac{\lambda}{z}\partial_q(z^2\mathcal{N}^{n-1}_{\lambda,q}(z))\\ &= \frac{1}{z} + \sum_{k=1}^{\infty}\sigma^n_q(k,\lambda)a_k z^k \ (n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}),\end{aligned}\tag{7}$$

where

$$\sigma^n_q(k,\lambda) = \left[1+\lambda\left([k+2]_q - 1\right)\right]^n.\tag{8}$$

Note that:

(i) $\lim_{q\to 1^-} N^n_{\lambda,q}(z) = D^{*n}_{\lambda}(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left[1+\lambda\,(k+1)\right]^n a_k z^k$;

(ii) $\lim_{q\to 1^-} N^n_{1,q}(z) = D^{*n}(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(k+2)^n a_k z^k$ (see [5] and [3, with $p = 1$]).

For $0<q<1,\ c\in(0,1]$ and $\tau\in C^*=C\backslash\{0\}$, let

$$\Sigma\mathcal{Q}_q^*(\tau,c)=\left\{\in\Sigma:1-\frac{1}{\tau}\left[\frac{z\partial_q((z))}{(z)}+\frac{1}{q}\right]\prec\triangle_c(z)\right\}, \tag{9}$$

and

$$\Sigma\mathcal{K}_q(\tau,c)=\left\{\in\Sigma:1-\frac{1}{\tau}\left[\frac{\partial_q(z\partial_q((z)))}{\partial_q((z))}+\frac{1}{q}\right]\prec\triangle_c(z)\right\}, \tag{10}$$

where $\triangle_c(z)=\sqrt{1+cz}$ $(z\in U)$ and $\prec$ denotes subordination (see [6, 8]).

From (1.9) and (1.10), we can conclude that

$$(z)\in\Sigma\mathcal{K}_q(\tau,c)\Leftrightarrow-qz\partial_q((z))\in\Sigma\mathcal{Q}_q^*(\tau,c). \tag{11}$$

By using the operator $N_{\lambda,q}^n(z)$ defined by (1.7), we introduce the classes $Q_q^{n*}(\tau,\lambda,c)$ and $K_q^n(\tau,\lambda,c)$ as follow:

For $n\in\mathbb{N}_0,\ 0<q<1,\ c\in(0,1],\ \lambda\geq0,\ \tau\in\mathbb{C}^*$, let

$$\mathcal{Q}_q^{n*}(\tau,\lambda,c)=\left\{\in\Sigma:\mathcal{N}_{\lambda,q}^n(z)\in\Sigma\mathcal{Q}_q^*(\tau,c)\right\}, \tag{12}$$

and

$$\mathcal{K}_q^n(\tau,\lambda,c)=\left\{\in\Sigma:\mathcal{N}_{\lambda,q}^n(z)\in\Sigma\mathcal{K}_q(\tau,c)\right\}, \tag{13}$$

where $\Sigma Q_q^*(\tau,c)$ and $\Sigma K_q(\tau,c)$ are given by (1.9) and (1.10).

From (1.12) and (1.13), we can conclude that

$$\mathcal{F}(z)\in\mathcal{K}_q^n(\tau,\lambda,c)\Leftrightarrow-qz\partial_q(\mathcal{F}(z))\in\mathcal{Q}_q^{n*}(\tau,\lambda,c). \tag{14}$$

For functions $(z)\in\Sigma$ given by (1.1) and $g(z)\in\Sigma$ given by

$$g(z)=z^{-1}+_{k=1}^{\infty}b_kz^k,$$

the Hadamard product (or convolution) is

$$(\ast g)(z)=z^{-1}+_{k=1}^{\infty}a_kb_kz^k=(g\ast)(z).$$

Since

$$\begin{aligned}-qz\partial_q(z)\ast g(z)&=(z^{-1}+_{k=0}^{\infty}-q[k]_qa_kz^k)\ast(z^{-1}+_{k=1}^{\infty}b_kz^k)=z^{-1}+_{k=0}^{\infty}-q[k]_qa_kb_kz^k,\\(z)\ast(-qz\partial_qg(z))&=(z^{-1}+\sum_{k=1}^{\infty}a_kz^k)\ast(z^{-1}+_{k=0}^{\infty}-q[k]_qb_kz^k)=z^{-1}+_{k=0}^{\infty}-q[k]_qb_ka_kz^k,\end{aligned}$$

then

$$-qz\partial_q(z)\ast g(z)=(z)\ast(-qz\partial_qg(z)) \tag{15}$$

2 Main Results

Unless indecated, let $0 < q < 1$, $c \in (0, 1]$, $\tau \in \mathbb{C}^*$, $\lambda \geq 0$, $n \in \mathbb{N}_0$, $\theta \in [0, 2\pi)$, $z \in U^*$, $\in \Sigma$.

The function $\in \Sigma Q_q^* (\tau, c)$ iff

$$z\left[(z) * \frac{1 + [\Psi(\theta) - q]z}{z(1-z)\,(1-qz)}\right] \neq 0 \ \ (z \in \mathbb{U}^*)\,, \tag{16}$$

where

$$\Psi(\theta) = \frac{e^{-i\theta}(1 + \sqrt{1 + ce^{-i\theta}})}{c\tau q}. \tag{17}$$

Proof. For $\in \Sigma$, To prove (2.1), we write (1.9) as

$$-q\frac{z\partial_q(z)}{(z)} = 1 - \tau q(1 - \sqrt{1 + c\omega(z)}), \tag{18}$$

hence

$$z\left[-qz\partial_q(z) - \left(1 + \tau q(\sqrt{1 + c\omega(z)} - 1)\right)(z)\right] \neq 0. \tag{19}$$

Now from (1.6), we may write (2.4) as

$$z\left\{\left[(z) * \frac{1-(q+1)z}{z\,(1-z)\,(1-qz)}\right] - \left[1 + \tau q\left(\sqrt{1 + ce^{i\theta}} - 1\right)\right]\left[(z) * \frac{1}{z(1-z)}\right]\right\} \neq 0,$$

which is equivalent to

$$z\left[(z) * \frac{1 + (-q + \frac{1}{\tau q(\sqrt{1+ce^{i\theta}}-1)})z}{z\,(1-z)\,(1-qz)}\left[-\tau q(\sqrt{1 + ce^{i\theta}} - 1)\right]\right] \neq 0,$$

or

$$z\left[(z) * \frac{1 + (-q + \frac{e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})}{\tau qc})z}{z\,(1-z)\,(1-qz)}\right] \neq 0, \ z \in \mathbb{U}^*, \tag{20}$$

which represents (2.1).

Reversely, suppose that $\in \Sigma$ satisfying (2.1). Since (2.1) is equivalent to (2.4), then

$$-q\frac{z\partial_q(z)}{(z)} \neq 1 - \tau q(1 - \sqrt{1 + ce^{i\theta}}). \tag{21}$$

Assume

$$\Omega(z) = -q\frac{z\partial_q(z)}{(z)} \text{ and } \Pi(z) = 1 - \tau q(1 - \sqrt{1 + cz}),$$

the relation (2.6) shows that $\Omega(U^*) \cap \Pi(\partial U^*) = \emptyset$ and thus the simply-connected domain $\Omega(U^*)$ is included in a connected component of $\mathbb{C}\backslash\Pi(\partial U^*)$. From here and using the fact $\Omega(0)=\Pi(0)$ together with the univalence of the function Π, it follows that $\Omega(z)\prec\Pi(z)$, that is $\in\Sigma Q_q^*(\tau,c)$.

Let $\in\Sigma$. Then $\in\Sigma K_q(\tau,c)$ iff

$$z\left[(z)*\frac{1-[3]_q z-(\Psi(\theta)-q)(q+1)\,qz^2}{z(1-z)(1-qz)(1-q^2z)}\right]\neq 0\ \ (z\in\mathbb{U}^*)\,, \tag{22}$$

where $\Psi(\theta)$ is given by (2.2).

Proof. From (1.11), $\in\Sigma K_q(\tau,c)$ iff $-qz\partial_q(z)\in\Sigma Q_q^*(\tau,c)$. Then from Theorem 1, $-qz\partial_q(z)\in\Sigma Q_q^*(\tau,c)$ iff

$$z\left[-qz\partial_q(z)*g(z)\right]\neq 0, \tag{23}$$

where

$$g(z)=\frac{1+(\Psi(\theta)-q)\,z}{z\,(1-z)\,(1-qz)},$$

thus

$$\begin{aligned}\partial_q g(z) &= \frac{g(qz)-g(z)}{(q-1)\,z}\\ &= \frac{-1+[3]_q z+(\Psi(\theta)-q)\,(q+1)qz^2}{qz^2\,(1-z)\,(1-qz)(1-q^2z)},\end{aligned}$$

and therefore

$$-qz\partial_q g(z)=\frac{1-[3]_q z-(\Psi(\theta)-q)\,(q+1)qz^2}{z\,(1-z)\,(1-qz)(1-q^2z)}.$$

Using the above relation and (1.15), it is simple to check that (2.8) is identical to (2.7).

$If\in Q_q^{n*}(\tau,\lambda,c)$. Then

$$1+_{k=1}^{\infty}\left[1+\frac{e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{\tau qc}\right]\sigma_q^n(k,\lambda)a_k z^k\neq 0\quad (z\in\mathbb{U}^*)\,. \tag{24}$$

Proof. If $\in\Sigma$, from Theorem 1, we have $\in Q_q^{n*}(\tau,\lambda,c)$ iff

$$z\left[\mathcal{N}_{\lambda,q}^n(z)*\frac{1+[\Psi(\theta)-q]z}{z(1-z)\,(1-qz)}\right]\neq 0\ \ (z\in\mathbb{U}^*)\,, \tag{25}$$

where $\Psi(\theta)$ is given by (2.2). Since

$$\begin{aligned}\frac{1}{z\,(1-z)\,(1-qz)} &= z^{-1}(1+z+z^2+...)(1+qz+q^2z^2+...)\\ &= z^{-1}+[2]_q+[3]_q z+[4]_q z^2+.... \end{aligned}\tag{26}$$

Then

$$\begin{aligned}\frac{1+[\Psi(\theta)-q]z}{z(1-z)\,(1-qz)} &= z^{-1}+(1+\Psi(\theta))+(1+[2]_q\Psi(\theta))z+(1+[3]_q\Psi(\theta))z^2+...\\ &= z^{-1}+\sum_{k=1}^{\infty}(1+\Psi(\theta)[k]_q)z^{k-1}.\end{aligned}$$

Now a basic calculation shows that (2.1) is identical to (2.9). Thus, we have the theorem.

Let $(z)\in\Sigma$ satisfies

$$\sum_{k=1}^{\infty}\left[[k]_q\left(\left|1+\sqrt{1+ce^{i\theta}}\right|\right)+|\tau|\,qc\right]\sigma_q^n(k,\lambda)\,|a_k|\leq|\tau|\,qc. \tag{27}$$

Then $\in Q_q^{n*}(\tau,\lambda,c)$.

Proof. Since

$$\begin{aligned}&\left|1+\sum_{k=1}^{\infty}\left(\frac{\tau qc+e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{\tau qc}\right)\sigma_q^n(k,\lambda)a_kz^k\right|\\ \geq\;& 1-\left|\sum_{k=1}^{\infty}\left(\frac{\tau qc+e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{\tau qc}\right)\sigma_q^n(k,\lambda)a_kz^k\right|\\ \geq\;& 1-\sum_{k=1}^{\infty}\left(\frac{|\tau|\,qc+e^{-i\theta}(\left|1+\sqrt{1+ce^{i\theta}}\right|)[k]_q}{|\tau|\,qc}\right)\sigma_q^n(k,\lambda)\,|a_k|>0,\end{aligned}$$

then, (2.12) holds and our result follows from Theorem 3.

If $\in K_q^n(\tau,\lambda,c)$. Then

$$1+_{k=1}^{\infty}\left[\frac{\tau qc+e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{\tau qc}\right](1-[k]_q)\,\sigma_q^n(k,\lambda)a_kz^k\neq 0\quad(z\in\mathbb{U}^*)\,. \tag{28}$$

Proof. From Theorem 2, we have $\in K_q^n(\tau,\lambda,c)$ iff

$$z\left[\mathcal{N}_{\lambda,q}^n(z)*\frac{1-[3]_qz-(\Psi(\theta)-q)\,(q+1)\,qz^2}{z(1-z)\,(1-qz)\,(1-q^2z)}\right]\neq 0\;\;(z\in\mathbb{U}^*)\,, \tag{29}$$

where $\Psi(\theta)$ is given by (2.2). Since

$$\begin{aligned}\frac{1}{z\,(1-z)\,(1-qz)(1-q^2z)} &= z^{-1}(1+z+z^2+...)(1+qz+q^2z^2+...)(1+q^2z+q^4z^2+...)\\ &= z^{-1}+(1+q+q^2)+(1+q+2q^2+q^3+q^4)z\\ &\quad+(1+q+2q^2+2q^3+2q^4+q^5+q^6)z^2+....\end{aligned} \tag{30}$$

Then

$$\begin{aligned}\frac{1-[3]_q z-(\Psi(\theta)-q)(q+1)qz^2}{z(1-z)\,(1-qz)\,(1-q^2z)} &= z^{-1}+(1-[2]_q)(1+\Psi(\theta)[2]_q)z \\ &\quad +(1-[3]_q)(1+\Psi(\theta)[3]_q)z^2+... \\ &= z^{-1}+\sum_{k=1}^{\infty}(1+\Psi(\theta)[k]_q)(1-[k]_q)z^{k-1}.\end{aligned}$$

Now a basic calculation shows that (2.14) is identical to (2.13). Thus, the proof is completed.

Using similar arguments to those in the proof of Theorem 4, we can prove the next result.

Let $(z)\in\Sigma$ satisfies

$$\sum_{k=1}^{\infty}\left[[k]_q\left(\left|1+\sqrt{1+ce^{i\theta}}\right|\right)+|\tau|\,qc\right](1-[k]_q)\,\sigma_q^n(k,\lambda)\,|a_k|\leq|\tau|\,qc, \qquad (31)$$

then $K_q^n(\tau,\lambda,c)$.

3 Open Problem

The authors suggest to find necessary and sufficient conditions for coefficients of function to belonge to the following class

$$\mathcal{Q}_q^{n*}(\lambda,A,B)=\left\{\ \in\Sigma:\mathcal{N}_{\lambda,q}^n(z)\in\Sigma\mathcal{Q}_q^*(A,B)\right\}, \qquad (32)$$

where

$$\Sigma\mathcal{Q}_q^*(A,B)=\left\{\ \in\Sigma:\frac{-q\varsigma\partial_q(\mathcal{F}(z))}{\mathcal{F}(z)}\prec\frac{1+Az}{1+Bz}\right\}, \qquad (33)$$

where $0<q<1,\ -1\leq B<A\leq 1$ and $N_{\lambda,q}^n(z)$ is the $q-$ operator defined by (1.7).

4 Conclusion

By using principle of subordination and $q-$meromorphic differential operator, in this paper, we have defined classes of meromorphic functions and obtained necessary and sufficient conditions for coefficients of function to belonge to these classes.

References

[1] M. K. Aouf, On a certain class of meromorphic univalent functions with positive coefficients, Rend. Mat. Appl., 7 (11), no. 2, (1991), 209-219.

[2] M. K. Aouf and H. E. Darwish, Certain meromorphically starlike functions with positive and fixed second coefficients, Tr. J. of Mathematics., 21 (1997), 311- 316.

[3] M. K. Aouf and H. M. Hossen, New criteria for meromorphic $p-$valent starlike functions, Tsukuba J Math., 17(2) (1993), 481–486.

[4] M. K. Aouf, A. O. Mostafa and F. Y. Al-Quhali, Class of meromorphic univalent functions with positive coefficients associated with a generalized $q-$Sălăgean operator, J. Taib. Univ. Sci., 14, no. 1, (2020), 1544-1553.

[5] M. K. Aouf and T. M. Seoudy, Subordination and superordination of certain linear operator on meromorphic function, Ann Univ Mariae Curie-Sklodowska Sec. A., 64(1) (2010), 1–16.

[6] T. Bulboacă, Differential Subordinations and Superordinations, New Results, Cluj-Napoca, House of Scientific Boook Publ., 2005.

[7] J. Miller, Convex meromorphic mappings and related functions, Proc. Am. Math. Soc., 25 (1970), 220-228.

[8] S. Miller and S. Mocanu, Differential Subordinations. Theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math., no. 255, New York, Marcel Dekker Inc., 2000, 480 p. DOI: 10.1201/9781482289817.

[9] Ch. Pommerenke, Über einige klassen meromorpher schlichter funktionen, Math. Z., 78 (1962), 263-284.

[10] Ch. Pommerenke, On meromorphic starlike functions, Pac. J. Math., 13 (1963), 221-235.

[11] H. Tang, H. M. Zayed, A. O. Mostafa and M. K. Aouf, Fekete-Szegö problems for certain classes of meromorphic functions using $q-$derivative operator, J. Math. Resear. Appl., 38, no. 3, (2018), 236-246.

[12] B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, Bull. Austral. Math. Soc., 43 (1991), 137-140.