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Wave Equations coupled by Von Karman Models with Rotational Forces

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Abstract

The purpose of this paper is to establish the existence and the uniqueness of a weak solution for the global functional energy associated to the wave equations coupled with the model of the von Karman evolution with rotational inertia and clamped boundary conditions. The resulting system of nonlinear equations is solved by a numerical scheme based on the finite difference method.

Keywords: *Von Karman equation, nonlinear plates, wave equations, rotational inertia, non-coupled method, finite difference method.*
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1 Introduction and position of the problem

A dynamic von Karman equations with rotational inertia $\alpha > 0$,[2] describes the oscillations of a rectangular elastic thin plate. In nonlinear wave equations coupled with the von Karman model, we usually consider the case when the

plate is interact with an acoustic wave, whose interface is in a two-dimensional manifold. The model of the wave-structure interaction with clamped boundary conditions, for the displacement u and the Airy stress function ϕ, is well-known and can be formulated as follows, see [2] for instance.
Find $(u,\phi)\in L^2\big([0,T],H^2(\omega)\big)\times L^2\big([0,T],H_0^2(\omega)\big)$ such that

$$(\mathbb{P}_0)\begin{cases} u_{tt}-\alpha\Delta u_{tt}+\Delta^2u-[\phi+F_0,u]=p(x,t) & in\quad \omega\times[0,T],\\ u_{|t=0}=\varphi_0,\quad (u_t)_{|t=0}=\varphi_1 & in\quad \omega,\\ u=\partial_\nu u=0 & on\quad \Gamma\times[0,T],\end{cases}$$

and

$$(\mathbb{Q})\begin{cases}\Delta^2\phi+[u,u]=0 & in\quad \omega\times[0,T],\\ \phi=0,\ \partial_\nu\phi=0 & on\quad \Gamma\times[0,T].\end{cases}$$

Here ω denotes the surface plate, φ_0 and φ_1 refer to the initial data, $\partial_\nu u=\nabla u.\vec{n}$ is the normal derivative inside the boundary Γ, the parameter $\alpha>0$ takes into account the rotational inertial momenta of the elements of the shell/plate and $[.,.]$ stands for the so-called Monge-Ampère operator defined by, [2]:

$$[\phi,u]=\partial_{11}\phi\partial_{22}u+\partial_{11}u\partial_{22}\phi-2\partial_{12}\phi\partial_{12}u. \tag{1.1}$$

The dynamic plate, which is subject to an internal force F_0 and to a pressure $p(x,t)$, is interacted with an enclosed acoustic field filling a bounded domain Σ in $\mathbb{R}^3$.

The aerodynamically pressure $p(x,t)$ has the following form, see [2]:

$$p(x,t)=p_0(x_1,x_2)+\mu.(\psi_t)_{x_3=0},\ \text{with } x=(x_1,x_2,x_3),$$

where $p_0\in L^2(\omega)$ and $\mu>0$ is the intensity of the interaction between the chamber of wave and the plate. The acoustic velocity potential $\psi(x,t)$ satisfies the following vibration problem, [1]:

$$(\mathbb{P}_1)\begin{cases}\psi_{tt}=\Delta\psi & in\quad \Sigma\times[0,T],\\ \psi_{|t=0}=\psi^0,\quad (\psi_t)_{|t=0}=\psi^1 & in\quad \Sigma,\\ \partial_{x_3}\psi=u_t & on\quad \omega\times[0,T],\\ \partial_\nu\psi=0 & on\quad \partial\Sigma/\omega\times[0,T].\end{cases}$$

Here ω is a subset of the boundary $\partial\Sigma$ of Σ such that $x=(x_1,x_2,0)$ for any $x\in\omega\cap\partial\Sigma$, ψ^0 and ψ^1 are initial data and $T>0$ is a real number. As usual, we use the notation $u_t=\frac{\partial u}{\partial t}$ and $u_{tt}=\frac{\partial^2u}{\partial^2t}$, for the sake of simplicity.

In [2], Chueshov and Lasiecka studied the problem of wave-plate interaction and established the existence and the uniqueness of a weak solution by using the theory of nonlinear semi-group. To justify the uniqueness, they used the weak continuity of nonlinear terms involving Airy stress function.

The aim in this paper is establish the existence and the uniqueness of the weak solution for the model of the wave-plate interaction, with rotational inertia and clamped boundary conditions. Our approach is based on an iterative problem whose the sequence-solution converges to the unique solution of the considered problem. Finally, we apply the finite difference method developed by Bilbao in [1] for illustrating the theoretical study.

2 Preliminary results

Throughout the following, Σ denotes a nonempty bounded domain in $\mathbb{R}^3$, with boundary $\partial\Sigma$, and ω is a nonempty bounded open domain in $\mathbb{R}^2$, with $\omega \subset \partial\Sigma$. We assume that $\Gamma := \partial\omega$ is regular and any point $x \in \partial\Sigma \cap \omega$ can be written in the form $x = (x_1, x_2, 0)$.

Let $p \geq 1$ be a real number and $m \geq 1$ be an integer. We denote by $|.|_{p,\omega}$ the standard norm of $L^p(\omega)$ and by $\|.\|_{m,\omega}$ the classical norm of $H^m(\omega)$.

For $u \in H_0^2(\omega)$ we set

$$\|u\| =: |\Delta u|_{2,\omega} = \Big(\int_\omega (\Delta u)^2\Big)^{\frac{1}{2}},$$

which defines a norm in the space $H_0^2(\omega)$, see [4, 7] for instance. For any fixed $\alpha > 0$ we also set:

$$\|u\|_\alpha = \|u\| + \alpha\, |\nabla u|_{2,\omega}^2 + |u_t|_{2,\omega}^2\,.$$

We define

$$W(0,T) = \Big\{u: \ u \in L^2\Big(\left[0,T\right], H_0^2(\omega)\Big), u_t \in L^2\Big(\left[0,T\right], L^2(\omega)\Big)\Big\},$$

which is a Hilbert space with the associated norm

$$\Big(|u|^2_{L^2\left([0,T],H_0^2(\omega)\right)} + |u_t|^2_{L^2\left([0,T],L^2(\omega)\right)}\Big)^{1/2}.$$

We recall the following results, see [4, 7] for instance.

Theorem 2.1. *Let $f \in L^2(\omega)$. Then the following problem*

$$(Q)\begin{cases} \Delta^2 v = f & in \quad \omega,\\ v = 0 & on \quad \Gamma,\\ \partial_\nu v = 0 & on \quad \Gamma,\end{cases}$$

has one and only one solution $v \in H_0^2(\omega) \cap H^4(\omega)$ satisfying

$$\|v\| \leq c_0\, |f|_{1,\omega}\,,$$

for some constant $c_0 > 0$ depending only on $mes(\omega)$.

Remark 2.2. *If the function* f *belongs to* $L^2\big([0,T], L^2(\omega)\big)$ *then the unique solution* v *of the problem* (Q) *satisfies* $v \in L^2\big([0,T], H_0^2(\omega) \cap H^4(\omega)\big)$.

Theorem 2.3. *Let* $g \in L^2\big([0,T], L^2(\omega)\big)$, $\psi^0 \in L^2\big([0,T], H^1(\Sigma)\big)$ *and* $\psi^1 \in L^2\big([0,T], L^2(\Sigma)\big)$. *Then the following vibration problem:*

$$(R)\begin{cases} \psi_{tt} = \Delta\psi & in \quad \Sigma \times [0,T], \\ \psi_{|t=0} = \psi^0_{|t=0}, \quad (\psi_t)_{|t=0} = \psi^1_{|t=0} & in \quad \Sigma, \\ \partial_{x_3}\psi = g & on \quad \omega \times [0,T], \\ \partial_\nu\psi = 0 & on \quad \partial\Sigma/\omega \times [0,T], \end{cases}$$

has one and only one solution $\psi \in H^1(\Sigma)$ *satisfying* $\psi_t \in L^2(\Sigma)$.

We mention that the solution ψ of the problem (R), as well as the following expressions

$$|\nabla\psi|^2_{2,\Sigma} =: \int_\Sigma \Big(\nabla\psi(x,y,z,t)\Big)^2 dxdydz, \quad |(\psi)_t|^2_{2,\Sigma} =: \int_\Sigma \Big(\frac{d}{dt}\psi(x,y,z,t)\Big)^2 dxdydz,$$

do not depend on $t \in [0,T]$.

We recall the following result as well, see [9].

Theorem 2.4. *Let* $g \in L^2\big([0,T], L^2(\omega)\big)$, $\psi^0 \in L^2\big([0,T], H^1(\Sigma)\big)$ *and* $\psi^1 \in L^2\big([0,T], L^2(\Sigma)\big)$. *Then the solution* ψ *of the problem* (R) *is such that* $\psi \in L^2\big([0,T], H^1(\Sigma)\big)$ *and* $\psi_t \in L^2\big([0,T], L^2(\Sigma)\big)$. *Moreover we have the following estimation*

$$\forall t \geq 0 \qquad |\psi_t|^2_{2,\Sigma} + |\nabla\psi|^2_{2,\Sigma} \leq e^T\Big(\big|\nabla\psi^0\big|^2_{2,\Sigma} + \big|\psi^1\big|^2_{2,\Sigma} + \int_0^T (|g|_{2,\omega})^2\Big).$$

Now, we are in the position to state our main result of this section as recited in the following.

Theorem 2.5. *Let* $f \in L^2\big([0,T], L^2(\omega)\big)$, $(\varphi_0, \varphi_1) \in H_0^2(\omega) \times H_0^1(\omega)$ *and* $(\psi^0, \psi^1) \in H^1(\Sigma) \times L^2(\Sigma)$. *Then the following problem:*

$$(\mathbb{S})\begin{cases} u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u - \mu(\psi_t)_{|x_3=0} = f & in \quad \omega \times [0,T], \\ \psi_{tt} = \Delta\psi, & in \quad \Sigma \times [0,T] \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0,T], \\ u_{|t=0} = \varphi_0, \quad (u_t)_{|t=0} = \varphi_1 & in \quad \omega, \\ \psi_{|t=0} = \psi^0, \quad (\psi_t)_{|t=0} = \psi^1 & in \quad \Sigma, \\ \partial_{x_3}\psi = u_t & on \quad \omega \times [0,T], \\ \partial_\nu\psi = 0 & on \quad \partial\Sigma/\omega \times [0,T], \end{cases}$$

has one and only one weak solution u *such that*

$(u, u_t) \in C^0\big([0,T], H_0^2(\omega)\big) \times H_0^1(\omega)$ *and* $(\psi, \psi_t) \in C^0\big([0,T], H^1(\Sigma)\big) \times L^2(\Sigma)$.

Moreover we have the following estimation

$$\|u\|^2+|u_t|^2_{2,\omega}+\alpha\,|\nabla u_t|^2_{2,\omega}+\mu\,|\psi_t|^2_{2,\Sigma}+\mu\,|\nabla\psi|^2_{2,\Sigma}\le e^T\Big(\,\|\varphi_0\|^2+|\varphi_1|^2_{2,\omega}+\alpha\,|\nabla\varphi_1|^2_{2,\omega}$$
$$+\,\mu\,\left|\psi^1\right|^2_{2,\Sigma}+\mu\,\left|\nabla\psi^0\right|^2_{2,\Sigma}+\int_0^T(|f|_{2,\omega})^2\Big).\quad(2.1)$$

Proof

For proving this result, we will follow the same approach as in [2, 8]. We consider an approximate problem to $(\mathbb{S})$ and we then use the variational method involving the n-order approximate solution. Let $\{\ e_k, e_k^1\ \}$ be a basis in the space $H_0^2(\omega)\times H^1(\Sigma)$. We define an n-order Galerkin approximate solution to the problem $(\mathbb{S})$ with clamped boundary conditions on the interval $[\ 0,T\ ]$, as a function (u^n,ψ^n) such that:

$$u^n(t)=\sum_{k=1}^{n}h_k(t)e_k\ \ \textit{and}\ \psi^n(t)=\sum_{k=1}^{n}l_k(t)e_k^1\quad n=1,2,3,...,$$

where $(h_k(t),l_k(t))\in W^{2,\infty}(0,T;\mathbb{R})\times W^{1,\infty}(0,T;\mathbb{R})$. Then, ϕ^n is determined by u^n according to the problem $(\mathbb{Q})$, (u_{n0},ψ_{n0}), (u_{n1},ψ_{n1}) are chosen such that (u_{n0},ψ_{n0}) converges to (φ_0,ψ^0) in $L^2\big([0,T]\,,H_0^2(\omega)\times H^1(\Sigma)\big)$ and (u_{n1},ψ_{n1}) converges to (φ_1,ψ^1) in $L^2\big([0,T]\,,H_0^1(\omega)\times L^2(\Sigma)\big)$.

For this, we set $u^{nm}=:u^n-u^m$, $\psi^{nm}=:\psi^n-\psi^m$ and, $u_{nmk}=:u_{nk}-u_{mk}$, $\psi_{nmk}=:\psi_{nk}-\psi_{mk}$ for $k=0,1$. Writing the variational problem associated to $(\mathbb{S})$, we get:

$$\int_\omega u_{tt}^n u_t^{nm}+\alpha\int_\omega\nabla u_{tt}^n\nabla u_t^{nm}+\int_\omega\Delta u^n\Delta u_t^{nm}-\mu\int_\omega u_t^{nm}(\psi_t^n)_{|x_3=0}=\int_\omega f u_t^{nm}$$

and

$$\int_\omega u_{tt}^m u_t^{nm}+\alpha\int_\omega\nabla u_{tt}^m\nabla u_t^{nm}+\int_\omega\Delta u^m\Delta u_t^{nm}-\mu\int_\omega u_t^{nm}(\psi_t^m)_{|x_3=0}=\int_\omega f u_t^{nm}.$$

Let us remark that we have

$$\int_\omega u_{tt}^{nm}u_t^{nm}+\alpha\int_\omega\nabla u_{tt}^{nm}\nabla u_t^{nm}+\int_\omega\Delta u^{nm}\Delta u_t^{nm}-\mu\int_\omega u_t^{nm}(\psi_t^{nm})_{|x_3=0}=0,$$

and also

$$\int_\Sigma\psi_{tt}^n\psi_t^{nm}+\int_\Sigma\nabla\psi^n\nabla\psi_t^{nm}=-\int_\omega u_t^n\big(\psi_t^{nm}\big)_{|x_3=0}.$$

Hence

$$\int_\Sigma\psi_{tt}^m\psi_t^{nm}+\int_\Sigma\nabla\psi^m\nabla\psi_t^{nm}=-\int_\omega u_t^m\big(\psi_t^{nm}\big)_{|x_3=0},$$

and then

$$\int_\Sigma \psi_{tt}^{nm}\psi_t^{nm} + \int_\Sigma \nabla\psi^{nm}\nabla\psi_t^{nm} = -\int_\omega u_t^{nm}\big(\psi_t^{nm}\big)_{|x_3=0}.$$

It follows that we have

$$\frac{1}{2}\frac{d}{dt}\Big(|u_t^{nm}|_{2,\omega}^2 + \|u^{nm}\|^2 + \alpha\,|\nabla u_t^{nm}|_{2,\omega}^2 \Big) - \mu\int_\omega u_t^{nm}\big(\psi_t^{nm}\big)_{|x_3=0} = 0,$$

and

$$\frac{\mu}{2}\frac{d}{dt}\Big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2 \Big) = -\mu\int_\omega u_t^{nm}\big(\psi_t^{nm}\big)_{|x_3=0}.$$

We then infer that

$$\frac{d}{dt}\Big(|u_t^{nm}|_{2,\omega}^2 + \|u^{nm}\|^2 + \alpha\,|\nabla u_t^{nm}|_{2,\omega}^2 \Big) + \mu\frac{d}{dt}\Big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2 \Big) = 0.$$

Integrating this latter equality with respect to $t > 0$, using the fact that $u^n_{|t=0} = u_{n0}$, $(u_t^n)_{|t=0} = u_{n1}$, $\psi^n_{|t=0} = \psi_{n0}$ and $(\psi_t^n)_{|t=0} = \psi_{n1}$, we then get

$$\begin{aligned}\|u^{nm}\|_\alpha + \mu\Big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2 \Big) = {}& |u_{nm1}|_{2,\omega}^2 + \alpha\,|\nabla u_{nm1}|_{2,\omega}^2 + \|u_{nm0}\|^2 \\ & + \mu\,|\nabla\psi_{nm0}|_{2,\Sigma}^2 + \mu\,|\psi_{nm1}|_{2,\Sigma}^2\,.\end{aligned}$$

For $0 \le s \le t$ we have

$$\begin{aligned}\|u^{nm}\|_\alpha + \mu\big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2 \big) \le {}& |u_{nm1}|_{2,\omega}^2 + \alpha\,|\nabla u_{nm1}|_{2,\omega}^2 + \|u_{nm0}\|^2 \\ + \mu\,|\nabla\psi_{nm0}|_{2,\Sigma}^2 + \mu\,|\psi_{nm1}|_{2,\Sigma}^2 + & \int_0^t (\|u^{nm}\|_\alpha + \mu\Big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2)\Big).\end{aligned} \tag{2.2}$$

For the sake of simplicity we set, for $0 \le s \le t$,

$$I(s) = \|u^{nm}\|_\alpha + \mu\big(|\psi_t^{nm}|_{2,\Sigma}^2 + |\nabla\psi^{nm}|_{2,\Sigma}^2 \big).$$

Then (2.2) yields

$$\begin{aligned}e^{-s}\Big(I(s) - \int_0^s I(\sigma)d\sigma\Big) \le e^{-s}\Big(& |u_{nm1}|_{2,\omega}^2 + \alpha\,|\nabla u_{nm1}|_{2,\omega}^2 + \|u_{nm0}\|^2 \\ & + \mu\,|\nabla\psi_{nm0}|_{2,\Sigma}^2 + \mu\,|\psi_{nm1}|_{2,\Sigma}^2 \Big).\end{aligned}$$

It follows that

$$\begin{aligned}\tfrac{d}{ds}\Big(e^{-s}\int_0^s I(\sigma)d\sigma\Big) & = e^{-s}I(s) - e^{-s}\int_0^s I(\sigma)d\sigma = e^{-s}\Big(I(s) - \int_0^s I(\sigma)d\sigma\Big), \\ & \le e^{-s}\Big(|u_{nm1}|_{2,\omega}^2 + \alpha\,|\nabla u_{nm1}|_{2,\omega}^2 + \|u_{nm0}\|^2 \\ & \qquad + \mu\,|\nabla\psi_{nm0}|_{2,\Sigma}^2 + \mu\,|\psi_{nm1}|_{2,\Sigma}^2 \Big).\end{aligned}$$

Remark that the following expression

$$\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}=I(0),$$

does not depend on s, and therefore we can write

$$\begin{aligned}\int_0^t\frac{d}{ds}\Big(e^{-s}\int_0^s I(\sigma)d\sigma\Big)ds\leq\Big(\int_0^t e^{-s}ds\Big)\Big(&\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}\\&+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big),\end{aligned}$$

from which we deduce

$$\begin{aligned}e^{-t}\int_0^t I(\sigma)d\sigma\leq(1-e^{-t})\Big(&\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}\\&+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big).\end{aligned}$$

Since

$$\int_0^t\Big(\left\|u^{nm}\right\|_{\alpha}+\mu\big(\left|\psi_t^{nm}\right|_{2,\Sigma}^{2}+\left|\nabla\psi^{nm}\right|_{2,\Sigma}^{2}\big)\Big)=\int_0^t I(\sigma)d\sigma,$$

then we have

$$\begin{aligned}\int_0^t I(\sigma)d\sigma&\leq\tfrac{(1-e^{-t})}{e^{-t}}\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big),\\&\leq(e^{t}-1)\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big),\\&\leq(e^{T}-1)\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big).\end{aligned}$$

This, when combined with (2.2), yields

$$\begin{aligned}\left\|u^{nm}\right\|_{\alpha}+\mu\Big(\left|\psi_t^{nm}\right|_{2,\Sigma}^{2}+\left|\nabla\psi^{nm}\right|_{2,\Sigma}^{2}\Big)&\leq\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}\\&\quad+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big)+(e^{T}-1)\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}\\&\quad+\left\|u_{nm0}\right\|^{2}+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big),\\&\leq e^{T}\Big(\left|u_{nm1}\right|_{2,\omega}^{2}+\alpha\left|\nabla u_{nm1}\right|_{2,\omega}^{2}+\left\|u_{nm0}\right\|^{2}\\&\quad+\mu\left|\nabla\psi_{nm0}\right|_{2,\Sigma}^{2}+\mu\left|\psi_{nm1}\right|_{2,\Sigma}^{2}\Big).\end{aligned}$$

The sequence $\big(u^n,\psi^n\big)$ is a Cauchy sequence and according to the n-order approximate problem we have $\big(u^n,\psi^n\big)\in C^0\big([0,T],H_0^2(\omega)\times H^1(\Sigma)\big)$ and $\big(u_t^n,\psi_t^n\big)\in C^0\big([0,T],H_0^1(\omega)\times L^2(\Sigma)\big)$. We then deduce that $\big(u^n,\psi^n\big)$ converges to (u,ψ) in the Banach space $C^0\big([0,T],H_0^2(\omega)\times H^1(\Sigma)\big)$ and $\big((u^n)_t,\psi_t^n\big)$ converges weakly to $\big((u)_t,\psi_t\big)$ in $C^0\big([0,T],H_0^1(\omega)\times L^2(\Sigma)\big)$.

For showing that (u,ψ) is a weak solution of the problem $(\mathbb{S})$, we proceed in an analogous way as in [8].

Let $\varphi_j \in C^1(0,T)$, $1 \leq j \leq j_0$, be such that $\varphi_j(T)=0$ and

$$\theta =: \sum_{j=1}^{j_0} \varphi_j \otimes e_j, \quad \varphi =: \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1.$$

With this, we have the following variational equations

$$-\int_0^T \int_\omega u_t^n \theta_t + \alpha \int_0^T \int_\omega \nabla u_t^n \nabla \theta_t - \mu \int_0^T \int_\omega \theta \big(\psi_t^n\big)_{|x_3=0} + \int_0^T \int_\omega \Delta u^n \Delta \theta$$

$$= \int_0^T \int_\omega f\theta - \int_\omega u_{n1}\theta(0) - \alpha \int_\omega \nabla u_{n1} \nabla \theta(0) \tag{2.3}$$

and

$$-\int_0^T \int_\Sigma \psi^n \varphi_t + \int_\Sigma \nabla \psi^n \nabla \varphi = -\int_\omega u_t^n \varphi + \int_\Sigma \psi_{n1}\varphi(0). \tag{2.4}$$

Letting $n \to +\infty$ in (2.3) and in (2.4), we therefore deduce that, for all $\theta \in L^2\big([0,T], H_0^2(\omega)\big)$, $\theta_t \in L^2\big([0,T], H^1(\omega)\big)$, $\varphi \in L^2\big([0,T], H^1(\Sigma)\big)$ and $\varphi_t \in L^2\big([0,T], L^2(\Sigma)\big)$ such that $\psi(T)=\varphi(T)=0$, we have

$$-\int_0^T \int_\omega u_t \theta_t + \alpha \int_0^T \int_\omega \nabla u_t \nabla \theta_t - \mu \int_0^T \int_\omega \theta \big(\psi_t\big)_{|x_3=0} + \int_0^T \int_\omega \Delta u \Delta \theta$$

$$= \int_0^T \int_\omega f\theta - \int_\omega \varphi_1 \theta(0) - \alpha \int_\omega \nabla \varphi_1 \nabla \theta(0)$$

and

$$-\int_0^T \int_\Sigma \psi \varphi_t + \int_\Sigma \nabla \psi \nabla \varphi = -\int_\omega u_t \varphi + \int_\Sigma \psi^1 \varphi(0).$$

We then conclude that (u,ψ) is a weak solution of the problem $(\mathbb{S})$.

Otherwise, by analogous method as in the proof of Theorem 2.4, it is not hard to show the following inequality

$$\|u\|^2 + |u_t|_{2,\omega}^2 + \alpha\,|\nabla u_t|_{2,\omega}^2 + \mu\,|\psi_t|_{2,\Sigma}^2 + \mu\,|\nabla\psi|_{2,\Sigma}^2 \leq e^T \Big(\|\varphi_0\|^2 + |\varphi_1|_{2,\omega}^2 + \alpha\,|\nabla\varphi_1|_{2,\omega}^2 + \mu\,|\psi^1|_{2,\Sigma}^2$$

$$+ \mu\,|\nabla\psi^0|_{2,\Sigma}^2 + \int_0^T \big(|f|_{2,\omega}\big)^2 \Big).$$

Now, we will establish the uniqueness. Let (u_1,ψ_1) and (u_2,ψ_2) be two solutions of the problem $(\mathbb{S})$. Then $\big(u_1-u_2, \psi_1-\psi_2\big)$ satisfies the following equations

$$\begin{cases} (1-\alpha\Delta)(u_1-u_2)_{tt}+\Delta^2(u_1-u_2)-\mu\big((\psi_1)_t-(\psi_2)_t\big)_{|x_3=0}=0 & in \quad \omega\times[0,T]\,, \\ (\psi_1-\psi_2)_{tt}+\Delta(\psi_1-\psi_2)=0 & in \quad \Sigma\times[0,T]\,, \\ u_1-u_2=\partial_\nu(u_1-u_2)=0 & on \quad \Gamma\times[0,T]\,, \\ (u_1-u_2)_{|t=0}=0, ((u_1-u_2)_t)_{|t=0}=0, & in \quad \omega, \\ (\psi_1-\psi_2)_{|t=0}=0, ((\psi_1-\psi_2)_t)_{|t=0}=0, & in \quad \Sigma, \\ \partial_{x_3}(\psi_1-\psi_2)=(u_1)_t-u_2)_t, & on \quad \omega\times[0,T]\,, \\ \partial_\nu(\psi_1-\psi_2)=0, & on \quad \partial\Sigma/\omega\times[0,T]\,. \end{cases}$$

It follows that

$$\begin{aligned} &\|u_1-u_2\|_\alpha+\mu\,|(\psi_1-\psi_2)_t\,|^2_{2,\omega}+\mu\,|\nabla(\psi_1-\psi_2)|^2_{2,\omega} \\ &\quad\le e^T\Big(\,\big|(u_1)^1-(u_2)^1\big|^2_{2,\omega}+\|(u_1)_0-(u_2)_0\|^2+\alpha\,\big|\nabla((u_1)^1-(u_2)^1)\big|^2_{2,\omega} \\ &\qquad\qquad+\mu\,\big|(\psi_1)^0-(\psi_2)^0\big|^2_{2,\omega}+\mu\,\big|(\nabla\psi_1)^0-(\nabla\psi_2)^0\big|^2_{2,\omega}\Big). \end{aligned}$$

Hence $u_1=u_2$ and $\psi_1=\psi_2$. The proof is finished. ±
For the sake of simplicity, we need to introduce the following notation

$$F_1(u,\phi)=:[\,\phi+F_0,u\,]+p. \tag{2.5}$$

We have the two following results as well.

Proposition 2.6. *Let* $\widetilde{u}=(u,\phi)$ *and* $\widetilde{v}=(v,\varphi)$ *be in* $L^2\big([0,T]\,,(H_0^2(\omega))^2\big)$ *such that*

$$\|\widetilde{u}\|_{(H_0^2(\omega))^2}\le c \quad and \quad \|\widetilde{v}\|_{(H_0^2(\omega))^2}\le c,$$

for some $c>0$ *small enough. If* $\|F_0\|_{2,\omega}$ *is small enough then the following inequality*

$$\|F_1(\widetilde{u})-F_1(\widetilde{v})\|_{L^1(\omega)}\le c_2\,\|(u,\phi)-(v,\varphi)\|_{(H_0^2(\omega))^2}$$

holds true for some $0<c_2<1$.

Proof

By (2.5) and the fact that the map $(\phi,u)\longmapsto[\phi,u]$ is bilinear we can write

$$\begin{aligned} \|F_1(\widetilde{u})-F_1(\widetilde{v})\|_{L^1(\omega)} &\le \Big|\,[\phi+F_0,u]-[\varphi+F_0,v]\,\Big|_{1,\omega} \\ &\le \Big|\,[\phi,u]-[\varphi,v]\,\Big|_{1,\omega}+\Big|\,[F_0,u-v]\,\Big|_{1,\omega}. \end{aligned} \tag{2.6}$$

It is easy to see that

$$\Big|\,[\phi,u]-[\varphi,v]\,\Big|_{1,\omega}=\Big|\,[\phi-\varphi,u]+[\varphi,u-v]\,\Big|_{1,\omega}\le\Big|\,[\phi-\varphi,u]\,\Big|_{1,\omega}+\Big|\,[\varphi,u-v]\,\Big|_{1,\omega}.$$

By the definition (1.1) of $[.,.]$ and using the Hölder inequality we get

$$
\begin{aligned}
&\left|[\phi-\varphi,u]\right|_{1,\omega} \leq \int_{\omega} |\partial_{11}(\phi-\varphi)|\,|\partial_{22}u| + \int_{\omega} |\partial_{22}(\phi-\varphi)|\,|\partial_{11}u| + 2\int_{\omega} |\partial_{12}(\phi-\varphi)|\,|\partial_{12}u| \\
&\leq |\partial_{11}(\phi-\varphi)|_{2,\omega}\,\|u\|_{2,\omega} + |\partial_{22}(\psi-\varphi)|_{2,\omega}\,\|u\|_{2,\omega} + 2\,|\partial_{12}(\phi-\varphi)|_{2,\omega}\,\|u\|_{2,\omega} \\
&\leq \|\phi-\varphi\|\,\|u\|_{2,\omega} + \|\phi-\varphi\|\,\|u\|_{2,\omega} + 2\,\|\phi-\varphi\|\,\|u\|_{2,\omega} \leq 4c\,\|\phi-\varphi\|\,. \qquad (2.7)
\end{aligned}
$$

By similar arguments, we can prove that

$$\left|[\varphi,u-v]\right|_{1,\omega} \leq 4c\,\|u-v\|$$

and

$$\left|[F_0,u-v]\right|_{1,\omega} \leq 4\,\|F_0\|_{2,\omega}\,\|u-v\|\,.$$

Substituting these in (2.6), with a simple manipulation, we then get

$$\|F_1(\widetilde{u})-F_1(\widetilde{v})\|_{L^1(\omega)} \leq \big(8c+4\,\|F_0\|_{2,\omega}\big)\,\|(u,\phi)-(v,\varphi)\|_{(H_0^2(\omega))^2}\,.$$

If we put

$$c_2 = 8c+4\,\|F_0\|_{2,\omega}\,,$$

and we choose

$$\|F_0\|_{2,\omega} < 1/4, \quad 0 < c < \frac{1-4\,\|F_0\|_{2,\omega}}{8}$$

then we obtain

$$\|F_1(\widetilde{u})-F_1(\widetilde{v})\|_{L^1(\omega)} \leq c_2\,\|(u,\phi)-(v,\varphi)\|_{(H_0^2(\omega))^2}\,,$$

with $0 < c_2 < 1$. The proof is complete. $\mp$

Proposition 2.7. *Let* $u,v \in H_0^2(\omega)$ *and* $F_0 \in H^4(\omega)$ *be with small enough norms. Let* $\phi,\varphi \in H_0^2(\omega)$ *be, respectively, the solutions of the two following equations:*

$$\Delta^2\phi = -\,[u,u] \quad \text{and} \quad \Delta^2\varphi = -\,[v,v]\,.$$

Then the following estimation

$$\|F_1(u,\phi)-F_1(v,\varphi)\|_{L^2(\omega)} \leq c_1\,\|(u,\phi)-(v,\varphi)\|_{(H_0^2(\omega))^2}$$

holds for some $0 < c_1 < 1$.

Proof

Following [2], we have

$$\Big|\,[u,\phi]-[v,\varphi]\,\Big|_{2,\omega} \leq c_0\big(\,\|u\|^2+\|v\|^2\,\big)\,\|u-v\|\,,$$

for some $c_0 > 0$. Let $c > 0$ be small enough, with $\|u\|_{2,\omega} \leq c$ and $\|v\|_{2,\omega} \leq c$. Then we have

$$\Big| [u, \phi] - [v, \varphi] \Big|_{2,\omega} \leq 2c_0c^2 \|u - v\|$$

and so

$$\begin{aligned} \|F_1(u, \phi) - F_1(v, \varphi)\|_{L^2(\omega)} &\leq \Big| [\phi + F_0, u] - [\varphi + F_0, v] \Big|_{2,\omega}, \\ &\leq \Big| [\phi, u] - [\varphi, v] \Big|_{2,\omega} + \Big| [F_0, u - v] \Big|_{2,\omega}, \\ &\leq \left(2c_0c^2 + 4 \|F_0\|_{4,\omega}\right) \|u - v\| . \end{aligned}$$

If we choose

$$\|F_0\|_{4,\omega} < \frac{1}{4} \quad \textit{and} \quad 0 < c < \sqrt{\frac{1 - 4 \|F_0\|_{4,\omega}}{2c_0}},$$

we have

$$0 < c_1 = 2c_0c^2 + 4 \|F_0\|_{4,\omega} < 1,$$

so concluding the proof. ±

3 Iterative approach

In this section, in order to establish the uniqueness of the solution (u, ϕ) satisfying $(\mathbb{P}_0)$ and $(\mathbb{Q})$, we will consider an iterative approach which we explain in the following.

Let $n \geq 2$ be an integer and let $u_1 \in H_0^2(\omega)$, with $u_1 \neq 0$. We construct $\phi_{n-1} \in H_0^2(\omega)$ as the solution of the equation $\Delta^2\phi_{n-1} = -[u_{n-1}, u_{n-1}]$, and (u_n, ψ_n) is such that:

$$(P_n) \begin{cases} (u_n)_{tt} - \alpha\Delta(u_n)_{tt} + \Delta^2 u_n - \mu\big((\psi_n)_t\big)_{|x_3=0} = F_1(u_{n-1}, \phi_{n-1}) & in \quad \omega \times [0, T]\,, \\ (\psi_n)_{tt} = \Delta\psi_n & in \quad \Sigma \times [0, T]\,, \\ u_n = \partial_\mu u_n = 0 & on \quad \Gamma \times [0, T]\,, \\ (u_n)_{|t=0} = \varphi_0, \big((u_n)_t\big)_{|t=0} = \varphi_1 & in \quad \omega, \\ (\psi_n)_{|t=0} = \psi^0, \quad \big((\psi_n)_t\big)_{|t=0} = \psi^1 & in \quad \Sigma, \\ \partial_{x_3}\psi_n = (u_n)_t & on \quad \omega \times [0, T]\,, \\ \partial_\nu\psi_n = 0 & on \quad \partial\Sigma/\omega \times [0, T]\,, \end{cases}$$

where $F_1(u, \phi)$ is defined by (2.5).

Let us explain more the use of the previous process. We start with given $0 \neq u_1$, we first find $\phi_1 \in H_0^2(\omega)$ solution of $\Delta^2\phi_1 = -[u_1, u_1]$. Then (u_2, ψ_2) is defined as solution of (P_2). We then pursue the same way for constructing ϕ_2, ψ_3, u_3, ϕ_3, ψ_4, u_4 in a similar manner.

We are now in the position to state the main result of this section that reads as follows.

Theorem 3.1. *Let* $p \in L^2(\omega)$, $\mu, \alpha > 0$, $(\varphi_0, \varphi_1) \in H_0^2(\omega) \times H_0^1(\omega)$ *and* $(\psi^0, \psi^1) \in H^1(\Sigma) \times L^2(\Sigma)$. *Assume that all the following quantities*

$$\|F_0\|_{4,\omega}\,,\ |p|_{2,\omega}\,,\ \|\varphi_0\|_{2,\omega}^2\,,\ |\varphi_1|_{2,\omega}^2\,,\ |\nabla\varphi_1|_{2,\omega}^2\,,\ \left|\nabla\psi^0\right|_{2,\Sigma}^2 \ \text{and}\ \left|\psi^1\right|_{2,\Sigma}^2$$

are small enough. Assume that further the initial data $u_1 \neq 0$ *is chosen independent in* t *and* $\|u_1\|$ *is small enough. Then there is one and only one weak solution* (u, ϕ, ψ) *in* $L^2\big([0,T]\,, H_0^2(\omega) \times H_0^2(\omega) \times H^1(\Sigma)\big)$ *satisfying* $\mathbb{P}_0, \mathbb{Q}, \mathbb{P}_1$, *and such that* $(u_t, \psi_t) \in L^2\big([0,T]\,, H^1(\omega) \times L^2(\Sigma)\big)$.

Proof We divide it into three steps.

Step 1: Let us consider the problems (P_n). Throughout this proof we will use the following notation

$$|||u||| := \|u\|^2 + |(u)_t|_{2,\omega}^2 + \alpha\,|\nabla(u)_t|_{2,\omega}^2 \ \text{ and }\ |||\psi|||_* =: |\psi_t|_{2,\Sigma}^2 + |\nabla\psi|_{2,\Sigma}^2\,.$$

By a mathematical induction on $n \geq 1$, we will prove that the following inequalities

$$|||u_n||| \leq \|u_1\|^2 \ \text{ and }\ \|\phi_n\|_{2,\omega} \leq \|u_1\|$$

hold true for all $n \geq 1$, with $0 \leq t \leq T$.

Since u_1 does not depend on t then, for $n = 1$ we have

$$|||u_1||| = \|u_1\|^2 + |(u_1)_t|_{2,\omega}^2 + \alpha\,|\nabla(u_1)_t|_{2,\omega}^2 = \|u_1\|^2\,.$$

Otherwise, ϕ_1 is the solution of $\Delta^2\phi_1 = -\,[\ u_1, u_1\ ]$, and consequently Theorem 2.1 ensures that there exists $c_0 > 0$ such that

$$\|\phi_1\| \leq c_0\,|[\ u_1, u_1\ ]|_{1,\omega}\,.$$

Using the proof of Proposition 2.6, with $\|u_1\| < c$ and $0 < 4c_0c < 1$, we can deduce that

$$\|\phi_1\| \leq 4c_0\,\|u_1\|^2 \leq 4c_0c\,\|u_1\| \leq \|u_1\|\,.$$

This means that the desired inequalities are true for $n = 1$.

Assume that for $k = 2, ..., n$ and $0 \leq t \leq T$, we have

$$|||u_k||| \leq \|u_1\|^2 \ \text{ and }\ \|\phi_k\|_{2,\omega} \leq \|u_1\|\,,$$

and according to Theorem (2.1), Proposition (2.6) we deduce that

$$\|\phi_n\| \leq c_0\,|[\ u_n, u_n\ ]|_{1,\omega} \leq 4c_0\,\|u_n\|^2\,.$$

Since (u_{n+1},ψ_{n+1}) is the solution of (P_{n+1}), Proposition 2.7, Theorem 2.1 and Theorem 2.5 imply that, there exists $0<c_1=:2c_0c^2+4\left\|F_0\right\|_{4,\omega}<1$, such that

$$\begin{aligned}
|||u_{n+1}|||+\mu\,|||\psi_{n+1}|||_* &\leq e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu(\left|\psi^1\right|^2_{2,\Sigma}+\left|\nabla\psi^0\right|^2_{2,\Sigma})\\
&\qquad+\int_0^T\left\|F_1(u_n,\phi_n)\right\|^2_{L^2(\omega)}\Big)\\
&\leq e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu(\left|\psi^1\right|^2_{2,\Sigma}+\left|\nabla\psi^0\right|^2_{2,\Sigma})+2\int_0^T\left|p\right|^2_{2,\omega}\\
&\qquad+\int_0^T 2c_1\left\|(u_n,\phi_n)\right\|^2_{(H^2_0(\omega))^2}\Big)\\
&\leq e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu(\left|\psi^1\right|^2_{2,\Sigma}+\left|\nabla\psi^0\right|^2_{2,\Sigma})+2T\left|p\right|^2_{2,\omega}\\
&\qquad+\int_0^T 2c_1\big(\left\|u_n\right\|^2+\left\|\phi_n\right\|^2\big)\Big)\\
&\leq e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu(\left|\psi^1\right|^2_{2,\Sigma}+\left|\nabla\psi^0\right|^2_{2,\Sigma})+2T\left|p\right|^2_{2,\omega}\\
&\qquad+\int_0^T 2c_1(1+4c_0)\left\|u_n\right\|^2\Big).
\end{aligned}$$

It follows that, with $\left\|u_n\right\|^2\leq\left\|u_1\right\|^2$, we have

$$\begin{aligned}
|||u_{n+1}|||+\mu\,|||\psi_{n+1}|||_* \;&\leq e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu\big(\left|\psi^1\right|^2_{2,\Sigma}+\left|\nabla\psi^0\right|^2_{2,\Sigma}\big)\\
&\quad+2T\left|p\right|^2_{2,\omega}\Big)+2Te^Tc_1(1+4c_0)\left\|u_1\right\|^2.
\end{aligned}$$

We can choose

$$0<c_3=2Te^Tc_1(1+4c_0)<1 \tag{3.1}$$

and

$$\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+2T\left|p\right|^2_{2,\omega}+\mu\Big(\left|\nabla\psi^0\right|^2_{2,\Sigma}+\left|\psi^1\right|^2_{2,\Sigma}\Big)\leq\frac{1-c_3}{e^T}\left\|u_1\right\|^2.$$

Thus, with $0<c_1<1$, we have

$$\begin{aligned}
|||u_{n+1}|||+\mu\,|||\psi_{n+1}|||_* \leq\;& e^T\Big(\left\|\varphi_0\right\|^2+\left|\varphi_1\right|^2_{2,\omega}+\alpha\left|\nabla\varphi_1\right|^2_{2,\omega}+\mu\big(\left|\nabla\psi^0\right|^2_{2,\Sigma}+\left|\psi^1\right|^2_{2,\Sigma}\big)\\
&+2T\left|p\right|^2_{2,\omega}\Big)+c_3\left\|u_1\right\|^2,\\
&\leq e^T\frac{1-c_3}{e^T}\left\|u_1\right\|^2+c_3\left\|u_1\right\|^2=\left\|u_1\right\|^2.
\end{aligned}$$

Furthermore, one has

$$\left\|\phi_{n+1}\right\|\leq c_0\left|[\,u_{n+1},u_{n+1}\,]\right|_{1,\omega},$$

which, with $\left\|u_1\right\|<c$ and $0<4c_0c<1$, immediately yields

$$\left\|\phi_{n+1}\right\|\leq 4c_0\left\|u_{n+1}\right\|^2\leq 4c_0\left\|u_1\right\|^2\leq 4c_0c\left\|u_1\right\|\leq\left\|u_1\right\|.$$

Summarizing, we have proved that, for all $n \geq 1$ and any $0 \leq t \leq T$,

$$|||u_n||| \leq \|u_1\|^2 \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\| .$$

In another part, we have

$$|||\psi_n|||_* \leq \frac{1}{\mu} \|u_1\|^2 .$$

Step 2: For $n \geq 2$, let (u_n, ψ_n) be the unique solution of (P_n) and let $2 \leq m \leq n$. Then, it is easy to see that $\psi_{nm} =: \psi_n - \psi_m$ and $u_{nm} =: u_n - u_m$ satisfy the following equations

$$\begin{cases} (u_{nm})_{tt} - \alpha\Delta(u_{nm})_{tt} + \Delta^2 u_{nm} - \mu((\psi_{nm})_t)_{|x_3=0} & \\ \qquad\qquad = F_1(u_{n-1}, \phi_{n-1}) - F_1(u_{m-1}, \phi_{m-1}) & \mathit{in} \quad \omega \times [0,T] , \\ (\psi_{nm})_{tt} = \Delta(\psi_{nm}) & \mathit{in} \quad \Sigma \times [0,T] , \\ u_{nm} = 0 \qquad \partial_\nu u_{nm} = 0 & \mathit{on} \quad \Gamma \times [0,T] , \\ (u_{nm})_{|t=0} = 0 \ , ((u_{nm})_t)_{|t=0} = 0 & \mathit{in} \quad \omega, \\ \partial_{x_3}(\psi_{nm}) = (u_{nm})_t & \mathit{on} \quad \omega \times [0,T] , \\ \partial_\nu(\psi_{nm}) = 0 & \mathit{on} \quad \partial\Sigma/\omega \times [0,T] , \\ (\psi_{nm})_{|t=0} = 0 \ , ((\psi_{nm})_t)_{|t=0} = 0 & \mathit{in} \quad \Sigma. \end{cases}$$

Note that $\phi_{n-1} - \phi_{m-1}$ is a solution of the following problem

$$\begin{cases} \Delta^2(\phi_{n-1} - \phi_{m-1}) = -[u_{n-1}, u_{n-1}] + [u_{m-1}, u_{m-1}] , & \mathit{in} \ \omega \times [0,T] \\ \\ \phi_{n-1} - \phi_{m-1} = \partial_\nu(\phi_{n-1} - \phi_{m-1}) = & \mathit{on} \ \ \Gamma \times [0,T] , \end{cases}$$

According to Theorem (2.1) we can write

$$\|\phi_{n-1} - \phi_{m-1}\| \leq c_0 \left|[u_{n-1}, u_{n-1}] - [u_{m-1}, u_{m-1}]\right|_{1,\omega} ,$$

and by Proposition 2.6, with (2.7), we infer that

$$\begin{aligned} \|\phi_{n-1} - \phi_{m-1}\| &\leq 4c_0\big(\|u_{n-1}\| + \|u_{m-1}\|\big) \|u_{n-1} - u_{m-1}\| \\ &\leq 8c_0 c \|u_{n-1} - u_{m-1}\| . \end{aligned} \tag{3.2}$$

Utilizing Proposition 2.7 and Theorem 2.5 again we get

$$\begin{aligned} |||u_n - u_m||| + \mu\, |||\psi_n - \psi_m|||_* &\leq e^T \int_0^T |F_1(u_{n-1}, \phi_{n-1}) - F_1(u_{m-1}, \phi_{m-1})|^2_{2,\omega} , \\ &\leq e^T c_1 \int_0^T \|(u_{n-1} - u_{m-1}), (\phi_{n-1} - \phi_{m-1})\|^2_{(H_0^2(\omega))^2} , \\ &\leq e^T c_1 \int_0^T \big(\|u_{n-1} - u_{m-1}\|^2 + \|\phi_{n-1} - \phi_{m-1}\|^2\big), \\ &\leq e^T c_1 \int_0^T \big(\|u_{n-1} - u_{m-1}\|^2 + 64c_0^2c^2 \|u_{n-1} - u_{m-1}\|^2\big), \\ &\leq e^T c_1(1 + 64c_0^2c^2) \int_0^T \|u_{n-1} - u_{m-1}\|^2 , \end{aligned}$$

We then get, with $c_4 = e^T c_1(1 + 64c_0^2 c^2)$,

$$\begin{aligned} |||u_n - u_m||| &\leq e^T c_1(1 + 64c_0^2 c^2) \int_0^T |||u_{n-1} - u_{m-1}||| \,, \\ &\leq (c_4)^{m-2} \int_0^T \dots \int_0^T |||u_{n-m+2} - u_1||| \,, \end{aligned}$$

and consequently

$$\begin{aligned} |||u_n - u_m||| &\leq (c_4)^{m-2} \textstyle\sum_{k=0}^{n-m+1} (c_4)^k \displaystyle\int_0^T \dots \int_0^T |||(u_2 - u_1)||| \,, \\ &\leq (c_4)^{m-2} \textstyle\sum_{k=0}^{n-m+1} (c_4)^k \big(2 \left\| u_1 \right\|^2 \big). \end{aligned}$$

This, with the help of (3.2), yields

$$\|\phi_n - \phi_m\| \leq 4c_0 \big(\|u_n\| + \|u_m\| \big) \|u_n - u_m\| \,.$$

The sequence $(u_n, \phi_{n-1})_{n\geq 2}$ is a Cauchy sequence in $L^2\big([0,T] \,, (H_0^2(\omega))^2 \big)$. It follows that (u_n, ϕ_{n-1}) converges to some (u, ϕ) in $L^2\big([0,T] \,, (H_0^2(\omega))^2 \big)$ and $((u_n)_t, \nabla (u_n)_t)$ converges to $((u)_t, \nabla (u)_t)$ in $L^2\big([0,T] \,, (L^2(\omega))^2 \big)$. We then infer that $(u_n)_t$ converges to u_t in $L^2([0,T] \,, L^2(\omega))$.

Furthermore, $(\nabla \psi_n, (\psi_n)_t)$ is also a Cauchy sequence in $L^2([0,T] \,, (L^2(\Sigma))^2)$ and therefore $(\nabla \psi_n, (\psi_n)_t)$ converges to some (ψ^*, ψ^{**}) in the Banach space $L^2\big([0,T] \,, (L^2(\Sigma))^2 \big)$. According to the Hölder inequality, we have

$$\begin{aligned} \left| \psi_n(t) - \psi_n(0) - \int_0^t \psi^{**} \right|_{2,\Sigma}^2 &= \int_\Sigma \left| \psi_n - \psi_n(0) - \int_0^t \psi^{**} \right|^2 = \int_\Sigma \left| \int_0^t \big((\psi_n)_t - \psi^{**} \big) \right|^2 , \\ &\leq \int_\Sigma T^2 \int_0^t |(\psi_n)_t - \psi^{**}|^2 \leq T^2 \int_0^t |(\psi_n)_t - \psi^{**}|_{2,\Sigma}^2 \,. \end{aligned}$$

Hence the function-sequence $\psi_n(t) - \psi_n(0)$ converges to $\displaystyle\int_0^t \psi^{**}$ in $L^2([0,T] \,, H^1(\Sigma))$.

According to Theorem 2.5, we have $\psi_n \in C^0\big([0,T] \,, H^1(\Sigma) \big)$. It follows that $(\psi_n)_t$ converges to $\psi_t = \psi^{**}$. Otherwise, by Proposition 2.7 we can say that $F_1(u_{n-1}, \phi_{n-1})$ converges to $F_1(u, \phi)$ in $L^2(\omega)$.

By the continuity of the operators "trace" and "∂_ν",

$$(u_n, \phi_{n-1})_\Gamma = (\partial_\nu u_n, \partial_\nu \phi_{n-1}) = (0,0) \ \text{ and } \ ((u_n)_t)_{|\Gamma} = 0$$

imply that

$$(u, \phi)_\Gamma = (\partial_\nu u, \partial_\nu \phi) = (0,0) \ \text{ and } \ (u_t)_{|\Gamma} = 0.$$

Thanks to Theorem 2.5, we can write

$$\big(u_n, (u_n)_t, \psi_n, (\psi_n)_t \big) \in C^0\big([0,T] \,, H_0^2(\omega) \times H^1(\omega) \times H^1(\Sigma) \times L^2(\Sigma) \big),$$

with

$$(u_n)_{|t=0} = \varphi_0, \quad ((u_n)_t)_{|t=0} = \varphi_1, \ (\psi_n)_{|t=0} = \psi^0, \quad ((\psi_n)_t)_{|t=0} = \psi^1$$

and so we infer that

$$u_{|t=0} = \varphi_0, \quad (u_t)_{|t=0} = \varphi_1, \ \psi_{|t=0} = \psi^0, \quad (\psi_t)_{|t=0} = \psi^1$$

To show that (u, ϕ, ψ) is a weak solution of $\mathbb{P}_0, \mathbb{Q}, \mathbb{P}_1$, we follow an analog method as that of [8]. Let $\phi_j \in C^1(0,T)$, $1 \le j \le j_0$, be such that $\varphi_j(T) = 0$, and

$$\theta = \sum_{j=1}^{j_0} \varphi_j \otimes e_j, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j^1,$$

where (e_k) and (e_k^1) are basis of $H_0^2(\omega)$ and $H^1(\Sigma)$, respectively. We have the following variational equations:

$$-\int_0^T \int_\omega u_t^n \theta_t + \alpha \int_0^T \int_\omega \nabla u_t^n \nabla \theta_t - \mu \int_0^T \int_\omega \theta (\psi_t^n)_{|x_3=0} + \int_0^T \int_\omega \Delta u^n \Delta \theta$$

$$= \int_0^T \int_\omega F_1(u_{n-1}, \phi_{n-1})\theta - \int_\omega u_{n1}\theta(0) - \alpha \int_\omega \nabla u_{n1} \nabla \theta(0) \tag{3.3}$$

and

$$-\int_0^T \int_\Sigma \psi^n \varphi_t + \int_\Sigma \nabla \psi^n \nabla \varphi = -\int_\omega u_t^n \varphi + \int_\Sigma \psi_{n1} \varphi(0). \tag{3.4}$$

Letting $n \to +\infty$ in (3.3) and (3.4) we deduce that, for all $\theta \in L^2([0,T], H_0^2(\omega))$, $\theta_t \in L^2([0,T], H^1(\omega))$, $\varphi \in L^2([0,T], H^1(\Sigma))$ and $\varphi_t \in L^2([0,T], L^2(\Sigma))$, such that $\theta(T) = \varphi(T) = 0$, we have

$$-\int_0^T \int_\omega u_t \theta_t + \alpha \int_0^T \int_\omega \nabla u_t \nabla \theta_t - \mu \int_0^T \int_\omega \theta (\psi_t)_{|x_3=0} + \int_0^T \int_\omega \Delta u \Delta \theta$$

$$= \int_0^T \int_\omega F_1(u, \phi)\theta - \int_\omega \varphi_1 \theta(0) - \alpha \int_\omega \nabla \varphi_1 \nabla \theta(0) \tag{3.5}$$

and

$$-\int_0^T \int_\Sigma \psi \varphi_t + \int_\Sigma \nabla \psi \nabla \varphi = -\int_\omega u_t \varphi + \int_\Sigma \psi^1 \varphi(0). \tag{3.6}$$

Summarizing, we have established that (u, ϕ) is a weak solution of the von Karman evolution and ψ is a solution to the vibration problem.

Step 3: We now prove the uniqueness. Assume that there exist two solutions (u^1, ϕ^1, ψ^1) and (u^2, ϕ^2, ψ^2) in $L^2\big([0,T], (H_0^2(\omega))^2 \times H^1(\Sigma)\big)$ such that, for some $c > 0$ small enough, we have $\|u^1\|^2 \leq c$ and $\|u^2\|^2 \leq c$. This implies that $u^{12} =: u^1 - u^2$ and $\psi^{12} =: \psi^1 - \psi^2$ satisfy the following equations

$$(\mathbb{P}_3)\left\{\begin{array}{ll}
u^{12}_{tt} - \alpha\Delta u^{12}_{tt} + \Delta^2 u^{12} - \mu(\psi^{12}_t)_{|x_3=0} & \\
\qquad\qquad\qquad\qquad = F_1(u^1,\phi_1) - F_1(u^2,\phi_2) & in \quad \omega\times[0,T]\,, \\
\psi^{12}_{tt} = \Delta\psi^{12}, & in \quad \Sigma\times[0,T]\,, \\
u^{12} = 0 \qquad ,\partial_\nu(u^{12}) = 0 & on \quad \Gamma\times[0,T]\,, \\
u^{12}(x_1,x_2,0) = 0, (u^{12})_t(x_1,x_2,0) = 0 & in \quad \omega, \\
\partial_\nu\psi^{12} = 0 & on \quad \Sigma/\omega\times[0,T]\,, \\
\partial_{x_3}\psi^{12} = u^{12}_t & on \quad \omega\times[0,T]\,, \\
\psi^{12}(x_1,x_2,0) = 0, (\psi^{12})_t(x_1,x_2,0) = 0 & in \quad \Sigma.
\end{array}\right.$$

This means that (u^{12},ψ^{12}) is a solution of the problem $(\mathbb{P}_3)$ and therefore, Theorem 2.5 and Proposition 2.7 ensure that we have

$$\begin{aligned}
&\left|\left\|u^1 - u^2\right\|\right| + \mu\left|\left\|\psi^1 - \psi^2\right\|\right|_* \leq e^T\int_0^T \left\|F_1(u^1,\phi^1) - F_1(u^2,\phi^2)\right\|^2_{L^2(\omega)} \\
&\leq e^T c_1\int_0^T (1 + 64c_0^2c^2)\left\|u^1 - u^2\right\|^2 \leq c_5\int_0^T \left|\left\|u^1 - u^2\right\|\right| + \mu\left|\left\|\psi^1 - \psi^2\right\|\right|_*
\end{aligned}$$

with $c_5 = e^Tc_1(1 + 64c_0^2c^2)$, and

$$\int_0^T \left|\left\|u^1 - u^2\right\|\right| + \mu\left|\left\|\psi^1 - \psi^2\right\|\right|_* \leq Tc_5\int_0^T \left|\left\|u^1 - u^2\right\|\right| + \mu\left|\left\|\psi^1 - \psi^2\right\|\right|_*.$$

If we chose $c > 0$ such that $c_5 = e^Tc_1(1+64c_0^2c^2) < \frac{1}{T}$ we then deduce that $u^1 = u^2$, $\psi^1 = \psi^2$ and $\phi^1 = \phi^2$. The proof of the theorem is completed. ±

We end this section by stating the following result.

Proposition 3.2. *Let* $(u,\psi) \in L^2\big([0,T]\,,(H^2_0(\omega))^2\times H^1(\Sigma)\big)$ *and* $\phi \in L^2\big([0,T]\,,H^2_0(\omega)\big)$ *be as above. Then the following equality*

$$\widetilde{E}\big(u(t),u_t(t),\phi\big)+\frac{\mu}{2}\big(\,|\psi_t|^2_{2,\Sigma}+|\nabla\psi|^2_{2,\Sigma}\,\big) = \widetilde{E}_1(\varphi_0,\varphi_1,\phi_0)+\frac{\mu}{2}\big(\,\left|\nabla\psi^0\right|^2_{2,\Sigma}+\left|\psi^1\right|^2_{2,\Sigma}\,\big),$$

holds for any $0 < t < T$, *where* $\phi_0 \in H^2_0(\omega)$ *is the unique solution of* $\Delta^2\phi_0 = -[\varphi_0,\varphi_0]$ *and,*

$$\widetilde{E}\big(u(t),u_t(t),\phi\big) =: \frac{1}{2}\big(\,|u_t|^2_{2,\omega}+\alpha\,|\nabla u_t|^2_{2,\omega}+\|u\|^2\,\big)+\frac{1}{4}\int_\omega \big(\,|\Delta\phi|^2-2\,[u,F_0]\,u-4pu\big),$$

$$\widetilde{E}_1(\varphi_0,\varphi_1,\phi_0) =: \frac{1}{2}\big(\,|\varphi_1|^2+\alpha\,|\nabla\varphi_1|^2_{2,\omega}+\|\varphi_0\|^2_{2,\omega}\big)+\frac{1}{4}\int_\omega(|\Delta\phi_0|^2-2\,[\varphi_0,F_0]\,\varphi_0-4p\varphi_0).$$

Proof

According to Theorem 2.5, for any $0 < t < t_0$ and $f = F_1(u,\phi)$, (u,ψ) satisfies the following variational equations

$$\int_\omega u_{tt}u_t + \alpha\int_\omega \nabla u_{tt}\nabla u_t + \int_\omega \Delta u\Delta u_t - \mu\int_\omega u_t(\psi_t)_{|x_3=0} = \int_\omega F_1(u,\phi)u_t,$$

and

$$\int_\Sigma \psi_{tt}\psi_t + \int_\Sigma \nabla\psi\nabla\psi_t = -\int_\omega u_t(\psi_t)_{|x_3=0}.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\Big(\,|u_t|^2_{2,\omega}+\alpha\,|\nabla u_t|^2_{2,\omega}+\|u\|^2\,\Big)+\frac{1}{2}\mu\frac{d}{dt}\Big(\,|\psi_t|^2_{2,\omega}+|\nabla\psi|^2_{2,\omega}\,\Big) = \int_\omega F_1(u,\phi)u_t, \tag{3.7}$$

with $F_1(u,\phi) =: [u,\phi+F_0]+p(x_1,x_2)$.

First we have

$$\int_0^t\int_\omega p(x_1,x_2)u_t = \int_\omega p(x_1,x_2)u(t) - \int_\omega p(x_1,x_2)u_0.$$

Otherwise, one has [2]

$$\begin{aligned}\int_0^t\int_\omega [u,\phi+F_0]\,u_t &= \int_0^t\int_\omega [u,\phi]\,u_t + \int_0^t\int_\omega [u,F_0]\,u_t,\\ &= \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}\big(\,[u,u]\,\big)\phi + \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}\big(\,[u,F_0]\,u\big),\\ &= -\frac{1}{4}\int_\omega |\Delta\phi|^2 + \frac{1}{4}\int_\omega |\Delta\phi_0|^2 + \frac{1}{2}\int_\omega [u,u]\,u - \frac{1}{2}\int_\omega [\varphi_0,\varphi_0]\,\varphi_0.\end{aligned}$$

If we integrate (3.7) with respect to t over $[0,T]$, we then get

$$\widetilde{E}(u(t),u_t(t),\phi)+\frac{\mu}{2}\Big(\,|\psi_t|^2_{2,\Sigma}+|\nabla\psi|^2_{2,\Sigma}\,\Big) = \widetilde{E}_1(\varphi_0,\varphi_1,\phi_0)+\frac{\mu}{2}\Big(\,\big|\psi^1\big|^2_{2,\Sigma}+\big|\nabla\psi^0\big|^2_{2,\Sigma}\,\Big),$$

completing the proof. ±

4 Numerical application

This section displays a numerical resolution of the previous theoretical study.

4.1 Preliminaries

Let Σ and ω be defined by

$$\Sigma =]0,1[\times]0,1[\times]0,1[\subset \mathbb{R}^3, \quad \omega =]0,1[\times]0,1[\subset \mathbb{R}^2$$

and $T > 0$. In order to solve numerically the problem $(\mathbb{P}_0)$, we introduce a uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with the internal points:

$$x_i = ih, \;\; y_j = jh, \quad i, j = 1, ...N-1, \;\; h = \frac{1}{N+1}, \;\; \Delta t = \frac{1}{T}.$$

Let $\overline{\omega}_h$ be the set of boundary mesh points and u_h be the finite-difference approximation of u.

In [1], Bilbao presented a numerical study of the convergence and stability of the conservative finite difference schemes for the dynamic von Karman plate equations via energy conserving methods.

For approaching the weak unique solution of the dynamic nonlinear plate coupled with wave equations, we will utilize the following discrete model of the von Karman evolution developed by Bilbao and Pereira in [1, 11].

$$(P_*)\begin{cases} \delta_t^2 u_{ij}^n - \alpha\delta_t^2(\delta_x^2+\delta_y^2)u_{ij}^n + \Delta_h^2 u_{ij}^n = [\; u_{ij}^n \;\; v_{ij}^n + F_{ij} \;] + p_{ij} + \mu.\delta_t \psi_{ij0}^n & in \quad \omega_h, \\ \Delta_h^2 v_{ij}^n = -[\; u_{ij}^n \;\; u_{ij}^n \;] & in \quad \omega_h, \\ u_{ij}^0 = (\varphi_0)_{ij}, \; \delta_t u_{ij}^0 = (\varphi_1)_{ij} & in \quad \omega_h, \\ u_{ij}^n = v_{ij}^n = 0 & on \quad \overline{\omega_h}, \\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & on \quad \overline{\omega_h}, \end{cases}$$

with the following discrete differential operators:

$$\delta_t^2 u_{ij}^n = \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2},$$

$$\delta_t u_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t},$$

$$\begin{aligned} \Delta_h^2 u_{ij}^n = h^{-4}\,[\; & u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j}) \\ & + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij} \;], \end{aligned}$$

$$\delta_x^2 u_{ij}^n = \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{(h)^2},$$

$$\delta_y^2 u_{ij}^n = \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2 u_{ij}^n = \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},$$

$$[\ u_{ij}^n,\ v_{ij}^n\ ] = \delta_x^2 u_{ij}^n \delta_y^2 v_{ij}^n - 2\delta_{xy}^2 u_{ij}^n \delta_{xy}^2 v_{ij}^n + \delta_y^2 u_{ij}^n \delta_x^2 v_{ij}^n.$$

We have transformed the above problem to the numerical resolution in three steps itemized as follows:
First step: We approach the velocity of potential gas to the following semi-linear wave equations by using the finite difference method of 7-point stencil, [6].

$$(P_{**}) \begin{cases} \psi_{tt} = \Delta\psi & in \quad \Sigma \times [0,T], \\ \psi_{|t=0} = \psi^0 \ \ (\psi_t)_{|t=0} = \psi^1 & in \quad \Sigma, \\ \partial_{x_3}\psi = u_t & on \quad \omega \times [0,T], \\ \partial_\nu \psi = 0 & on \quad \partial\Sigma/\omega \times [0,T], \end{cases}$$

Second step: We use the numerical procedure of 13-point formula of finite difference developed by Gubta in [5] for illustrating the weak solution of the following biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & in \quad \omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Third step: According to the first and second steps, we use the discrete model of the von Karman evolution (P_*) for illustrating the unique solution of the wave equations coupled with a dynamic von Karman evolution.

4.2 Non-coupled Approach

In [5], Gubta presented a numerical analysis of the finite-difference method for solving the Biharmonic equation. Such a method is known as the non-coupled method of 13-point. Moreover, Glowinski and Pironneau [4] made the observation that the 13-point finite difference scheme combined with a quadratic extrapolation formula near the boundary is equivalent to the mixed finite element method with piecewise linear elements.

Proposition 4.1. *[5], The 13-point approximation of the Biharmonic equation for approaching the unique solution v of the problem (P) is defined by:*

$$(1) \begin{cases} L_h v_{ij} = & h^{-4}\left[v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j})\right. \\ & \left. +2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}) - 20v_{ij}\right] = f_1(x_i, y_j) \end{cases}$$

for $i, j = 1, 2, ..., N-1$, where we set $v_{ij} = v(x_i, y_j)$.

When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$\begin{array}{l} v_{i-2,j} = \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i-1,j} \\ v_{i,j-2} = \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j-1} \\ v_{i+2,j} = \frac{1}{2}v_{i+1,j} - v_{ij} + \frac{3}{2}v_{i-1,j} - h(\partial_x v)_{i+1,j} \\ v_{i,j+2} = \frac{1}{2}v_{i,j+1} - v_{ij} + \frac{3}{2}v_{i,j-1} - h(\partial_y v)_{i,j+1} \end{array}$$

Remark 4.2. *Gubta in [5] generalized the approximation of $\partial_\nu v$ known as the (p, q) formula or the two-point formula. Otherwise, a detailed study of the linear matrix system of the scheme* (1) *can be found in [9, 10].*

4.3 Finite difference schemes

In this subsection, we investigate a finite difference method for approaching the $3-D$ wave equation by using 7-point stencil on the cubic lattice presented in [6]. We utilize the uniform mesh previously described. Let Σ_h be the set of all mesh points inside Σ with internal points

$$x_i = ih, \;\; y_j = jh, \;\; z_k = kh, \quad i, j, k = 1, ...N-1, \;\; h = \frac{1}{N+1}, \;\; \Delta t = \frac{1}{T}.$$

Let $\overline{\Sigma}_h$ be the set of boundary mesh points and ψ_h be the finite-difference approximation of ψ.

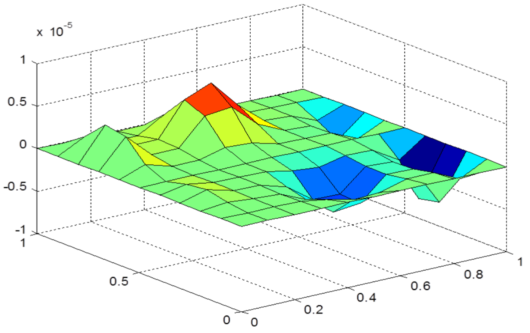
Let

$$\Delta x = \Delta y = \Delta z = \frac{1}{h}, \;\; h = \frac{1}{N+1}, \;\; \Delta t = \frac{1}{T}$$

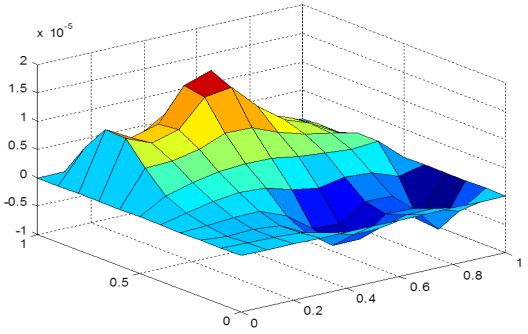
and we introduce the last typical notation for discrete differential operators. We approximate the problem (P_{**}) by the following finite difference system, [6].

$$(2) \left\{ \begin{array}{lll} \delta_t^2 \psi_{ijk}^n = \Delta_h \psi_{ijk}^n & in & \Sigma_h, \\ \Psi_{ijk}^0 = (\psi^0)_{ijk}, \; \delta_t \psi_{ijk}^0 = (\psi^1)_{ijk} & in & \Sigma_h, \\ (\partial_\nu \psi)_{ijk}^n = 0 & on & \overline{\Sigma}_h / \overline{\omega}_h, \\ \delta_z \psi_{ij0}^n = \delta_t u_{ij0}^n & on & \overline{\Sigma}_h \cap \overline{\omega}_h. \end{array} \right.$$

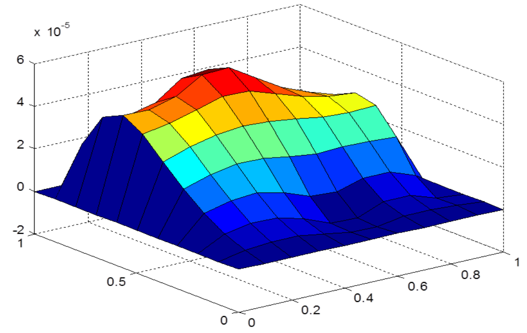
Example 1. Consider the following analytical body forces:
$d_1 = (sin(\pi x) sin(\pi y))^2, \;\; d_4 = (cos(2\pi x))^2, \;\; d_2 = (sin(\pi y))^2 cos(2\pi x)$
$d_3 = (sin(\pi x))^2 sin(2\pi y), \quad d_5 = (cos(2\pi y) sin(2\pi x))^2$
$p(x, y) = 2.10^{-8} \left[(2d_1 - 8(\pi)^2)(d_2 + d_3 - d_4) 8((\pi)^4 d_2 d_3 - d_5) - \mu d_1\right],$
$u_0 = 10^{-1} x^2 y^3 (e^{-x^2-y^2})(sin(2\pi x) cos(2\pi y))^2, \quad \psi_0 = 10^{-6} (xy)^2 (e^{-x^2-y^2}),$
$\psi_1 = 10^{-4} (xy)(e^{-x^2-y^2})(sin(\pi(x-y) + \pi))^2, \;\; u_1 = 15.10^{-4} x^2 y^3 x^2 (y-1)^3 e^{-x^2-y^2}$



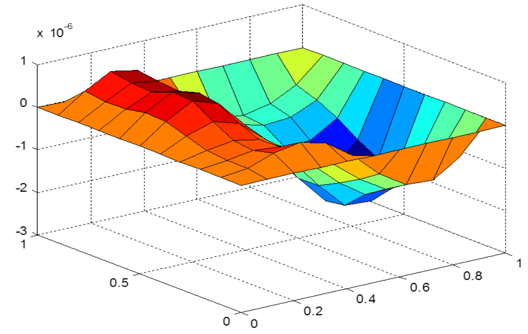
Displacement of plate, $T = 0.1s$



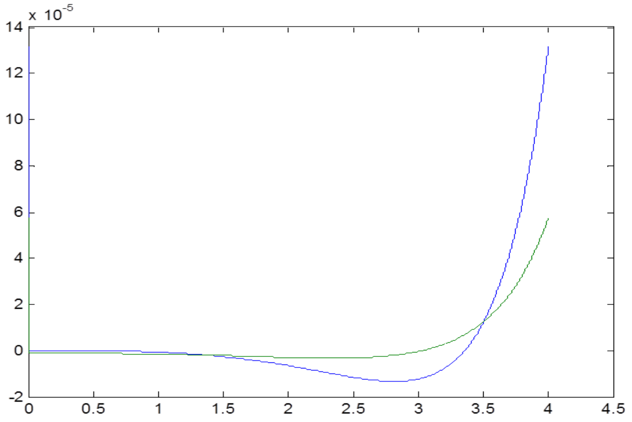
Displacement of plate, $T = 2.5s$



Displacement of plate, $T = 3s$



Velocity wave, $T = 1.5s, \ z = 0$



Displacement and wave in the point $(2,7)$ *for* $T = 4s$

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