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Notes on generalized weak demicompact operators

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Abstract

In this paper, we characterize generalized weak demicompact linear operators by means of the De Blasi measure of noncompactness. Moreover we investigate the left and right essential spectrum of the sum of two bounded operators acting on a Banach space. The results are formulated in terms of the DP property and some quantities related with the measure of weak noncompactness.

Keywords: *Generalized weakly demicompactness, Measure of weak noncompactness, Essential spectrum.*
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1 Introduction

In Fredholm theory, several analyzes of the essential spectrum of bounded or unbounded linear operators are based on the compactness or weak compactness classes [12, 13]. Recently, the sets of demicompact operators and theirs extensions appeared as large classes, containing compact operators, and played an important role in the investigation of essential spectra of linear operators acting on Banach spaces (see for instance [2, 5, 14, 15, 16, 17]). In particular, in 2019, I. Ferjani, A. Jeribi and B. Krichen in [9] introduced a new concept, related with Riesz operators, the so-called generalized weak demicompactness (for short GWDC). They provided some characterizations linking GWDC and upper-semi-Fredholm operators and studied some stability results for essential spectra. In [10], the same authors studied the relative generalized weak demicompactness with respect to a given unbounded linear operator as an extension of the GWDC class. This definition asserts that if T and S are two closed operators with domains satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, then T is called a generalized weakly S-demicompact operator if there exists a finite subset Θ of $\mathbb{C}$ containing 0 such that for all $\lambda \in \mathbb{C}\backslash\Theta$, $\frac{1}{\lambda}T$ is weakly S-demicompact operator, $T - \lambda S$ has a finite ascent and all $\lambda \in \sigma_S(T)\backslash\Theta$ are eigenvalues of finite multiplicities with no accumulation point except possibly points of Θ. Here, $\sigma_S(T) := \mathbb{C}\backslash \{\lambda \in \mathbb{C} : \lambda S - T$ has a bounded inverse $\}$, denotes the S-spectrum of T (see [8]). When $S = I$, $\sigma_S(T)$ will simply denoted by $\sigma(T)$.

Motivated by the analysis started in [10, 14, 15, 16, 17], we give some extended results to characterize GWDC operators by means of the De Blasi measure of noncompactness. The obtained results are used to investigate the left and right essential spectrum of the sum of two bounded operators.

In what follows, we present some notations and standard definitions from the Fredholm theory. Let X and Y be two infinite dimensional Banach spaces. The subsets $\mathcal{L}(X, Y)$, and $\mathcal{K}(X, Y)$ will respectively denote the set of all bounded and compact operators from X into Y. For $\mathcal{L}(X, Y)$, we denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, the null space and the range of T. We define the nullity $\alpha(T)$ (resp. the deficiency $\beta(T)$) as the dimension of $\mathcal{N}(T)$ in X (resp. the codimension of $\mathcal{R}(T)$ in Y). We denote by $asc(T)$ the ascent of T, i.e. the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$. Now we recall that an operator $T \in \mathcal{L}(X, Y)$ is weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$. The family of weakly compact linear operators from X into Y, is denoted by $\mathcal{W}(X, Y)$. The operator T is said a Dunford-Pettis operator (for short DP operator), if T maps weakly compact sets into compact ones. Recall also that T have a left Fredholm inverse (resp. a right Fredholm inverse) if there exists $T_l \in \mathcal{L}(Y, X)$ (resp. $T_r \in \mathcal{L}(Y, X)$) such that $I_X - T_l T \in \mathcal{K}(X)$ (resp. $I_Y - T T_r \in \mathcal{K}(Y)$). The set of upper semi-Fredholm and lower semi-Fredholm operators from X into Y are respectively defined by:

$$\Phi_+(X, Y) := \{T \in \mathcal{L}(X, Y) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\},$$

$$\Phi_-(X, Y) := \{T \in \mathcal{L}(X, Y) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

The set of Fredholm operators from X into Y is defined by:

$$\Phi(X, Y) := \Phi_-(X, Y) \cap \Phi_+(X, Y).$$

The sets of left and right Fredholm operators are respectively defined by:

$$\Phi_l(X, Y) := \{T \in \mathcal{L}(X, Y) : \mathcal{R}(T) \text{ is closed, complemented, and } \alpha(T) < \infty\},$$

$$\Phi_r(X, Y) := \{T \in \mathcal{L}(X, Y) : \mathcal{R}(T) \text{ is a closed, } \mathcal{N}(T) \text{ is complemented, and } \beta(T) < \infty\}.$$

The above sets satisfy the following inclusions:

$$\Phi(X,Y) \subset \Phi_l(X,Y) \text{ and } \Phi(X,Y) \subset \Phi_r(X,Y).$$

For an operator $T \in \Phi(X,Y)$, the index of T is defined by the number $i(T) = \alpha(T)-\beta(T)$. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_l(X,Y)$, and $\Phi_r(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_l(X)$, and $\Phi_r(X)$, respectively. For $X = Y$, a complex number λ is in Φ_T, Φ_{lT}, or Φ_{rT} if $\lambda - T$ is in $\Phi(X)$, $\Phi_l(X)$, or $\Phi_r(X)$ respectively.

Now a bounded linear operator F is called an upper semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ whenever $U \in \Phi_+(X,Y)$. We denote by $\mathcal{F}_+(X,Y)$ the set of upper semi-Fredholm perturbations. The sets of left Weyl and right Weyl operators are defined by:

$$\mathcal{W}_l(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is left Fredholm and } i(T) \leq 0\},$$

$$\mathcal{W}_r(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is right Fredholm and } i(T) \geq 0\}.$$

First, let us recall the axiomatic approach in defining measures of weak noncompactness. Let us denote by X a Banach space, by $\mathcal{M}_X$ the set of all bounded subset in X and by $\mathcal{W}_X$ the subfamily of $\mathcal{M}_X$ consisting of all relatively weakly compact sets.

Definition 1.1 [4] A measure of weak noncompactness on $\mathcal{M}_X$ (for short, MWNC) is a mapping $\nu : \mathcal{M}_X \to \mathbb{R}_0^+$ satisfying the following conditions. For all $A,\ B \in \mathcal{M}_X$:
(i) $\nu(B) = 0$ if, and only if, B is relatively weakly compact.
(ii) If $A \subset B$, then $\nu(A) \leq \nu(B)$.
(iii) $\nu(\overline{conv(B)}) = \nu(B)$. Here, $\overline{conv(B)}$ denotes the closed convex hull of B.
(iv) $\nu(A \cup B) = \max\{\nu(A), \nu(B)\}$.
(v) $\nu(\lambda A + B) \leq |\lambda|\nu(A) + \nu(B)$ for all $\lambda \in \mathbb{C}$.
(vi) (Cantor intersection property) If $(A_n)_n \subset \mathcal{M}_X$ such that $A_n = \overline{A}_n$ and $A_{n+1} \subset A_n$ for $n = 1, 2, ...$ and if $\lim_{n\to+\infty} \nu(A_n) = 0$, then $A_\infty = \bigcap_{n=1}^{+\infty} A_n \neq \emptyset$.

As an example of a measure of weak noncompactness in a Banach space, we cite the De Blasi measure ω defined on $\mathcal{M}_X$ as follows:

$$\omega(A) = \inf\{r > 0, \text{ there exists } N \in \mathcal{W}_X \text{ such that } A \subset N + \overline{B}_r\}.$$

This function has several useful properties (see [7]).

Definition 1.2 Let X and Y be two Banach spaces and let ω be the De Blasi MWNC in Y. We define the function

$$\begin{aligned} \psi_\omega :\ \mathcal{L}(X,Y) &\longrightarrow [0,+\infty[\\ T &\longmapsto \psi_\omega(T) = \omega(T(B_X)), \end{aligned}$$

ψ_ω is called a measure of weak noncompactness of operators associated to ω.

Resting upon this definition, we get the following properties of the function ψ_ω.

Proposition 1.1 [1] Let X and Y be be two Banach spaces and $T, S \in \mathcal{L}(X,Y)$. Let ω be the De Blasi MWNC in Y and ψ_ω be the MWNC of operators associated to ω. Then we have
(i) $\psi_\omega(T) = 0$ if, and only if, T is weakly compact.
(ii) $\psi_\omega(\lambda T + S) \leq |\lambda|\psi_\omega(T) + \psi_\omega(S)$ for all $\lambda \in \mathbb{C}$.
(iii) If $X = Y$, then $\psi_\omega(ST) \leq \psi_\omega(S)\psi_\omega(T)$.
(vi) If $K \in \mathcal{W}(X,Y)$, then $\psi_\omega(T + K) = \psi_\omega(T)$.

Several essential spectra are defined an studied in the litterature (see for example [12]). In this research work, we are basically interested in the following essential spectra:

$$
\begin{aligned}
\sigma_l(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \notin \Phi_l(X)\} \ : \ \text{the left Fredholm spectrum,}\\
\sigma_r(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \notin \Phi_r(X)\} \ : \ \text{the right Fredholm spectrum,}\\
\sigma_{ew}^l(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \notin \mathcal{W}_l(X)\} \ : \ \text{the left Weyl spectrum,}\\
\sigma_{ew}^r(T) &:= \{\lambda \in \mathbb{C} \ : \ \lambda - T \notin \mathcal{W}_r(X)\} \ : \ \text{the right Weyl spectrum.}
\end{aligned}
$$

This paper is divided into three sections. In the next section we establish the relationship between generalized weak demicompact operators and the De Blasi MWNC. In section 3 we determine the left and right essential spectrum of the sum of two bounded linear operators by means of the essential spectrum of each one.

2 Relationship between generalized weak demicompact operators and the De Blasi MWNC

Throughout this paper, we are working on two spaces, for example X and Y with their respective measures of weak noncompactness ω_X and ω_Y. However, and in order to simplify our reasoning, ω_X and ω_Y will be simply denoted $\omega(.)$. Of course, the reader will be able to link either to ω_X or to ω_Y. Furthermore, we denote $\to$ for the strong convergence (with respect to the norm of X) and $\rightharpoonup$ for the weak convergence (with respect to the weak topology of X).

Let us start by the following lemma which will be useful for some proofs. For this, let X and Y be two Banach spaces, $T \in \mathcal{L}(X,Y)$, ω a measure of weak noncompactness in X and Y and let ψ_ω be the measure of weak noncompactness of operators associated to ω.

Lemma 2.1 Let X and Y be two complex Banach spaces such that X is non reflexive and let $T \in \mathcal{L}(X,Y)$ such that $\mathcal{R}(T)$ is closed in Y. Assume that there exists $C \geq 0$ such that for all $x \in X$, $\|x\| \leq C\|Tx\|$. Then, for all $F \in \mathcal{M}_X$ we have

$$\omega(F) \leq C\,\omega(T(F)).$$

Proof 2.2 *For all $x \in X$, we have $\|x\| \leq C\|Tx\|$, then T is one to one. Since $\mathcal{R}(T)$ is a closed subspace of Y, it follows that T is boundedly invertible from X into $\mathcal{R}(T)$ and $\|T^{-1}\| \leq C$. Now, let $F \in \mathcal{M}_X$, then $T(F) \in \mathcal{M}_Y$ and we have*

$$\begin{aligned} \omega(F) &= \omega(T^{-1}(T(F))) \\ &\leq \psi_\omega(T^{-1})\,\omega(T(F)) \\ &\leq \omega(T^{-1}(\overline{B_Y}))\,\omega(T(F)) \\ &\leq w(T^{-1}(\overline{B_Y}))\,\omega(T(F)) \\ &\leq \|T^{-1}\|\,\omega(T(F)). \end{aligned}$$

Hence,

$$\omega(F) \leq C\;\omega(T(F)).$$

Which achieves the proof.

Theorem 2.3 Let X be a non reflexive Banach space and $A \in \mathcal{L}(X)$ such that A is a DP operator. Then, A is GWDC if, and only if, there exist a positive constant C and a finite subset Θ of $\mathbb{C}$ containing 0 such that for all bounded set $G \in \mathcal{M}_X$

$$\omega(G) \leq C\,\omega\left((\lambda I - A)(G)\right), \text{ for all } \lambda \in \mathbb{C}\backslash\Theta. \qquad \diamondsuit$$

Proof 2.4 *Suppose that A is GWDC, then by using Theorem 3.1 in [9], there exists a finite subset Θ of $\mathbb{C}$ containing 0 such that $\lambda I - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\Theta$. Now, since $asc(\lambda I - A)$ is finite, it follows that $i(\lambda I - A) \leq 0$, applying Lemma 3 in [18], there exist a compact operator K and a bounded below operator A_0 such that $\lambda I - A = K + A_0$. Since A_0 is bounded below, there exists a positive constant C such that for all $x \in X$.*

$$\|x\| \leq C\|A_0 x\|.$$

Hence, by applying Lemma 2.1, we deduce for any bounded set $G \subset X$ that

$$\omega(G) \leq C\;\omega\left((\lambda I - A)(G)\right).$$

To prove the converse, suppose that there exist a positive constant C and a finite subset Θ of $\mathbb{C}$ containing 0 such that for every bounded set G of X,

$$\omega(G) \leq C\;\omega\left((\lambda I - A)(G)\right), \textit{ for all } \lambda \in \mathbb{C}\backslash\Theta.$$

Let $(x_n)_n$ be a bounded sequence of X such that

$$(\lambda I - A)x_n \rightharpoonup x \in X.$$

Choose $G = \{x_n;\ n \in \mathbb{N}\}$. It is clear that G is bounded and $\omega\left((\lambda I - A)(G)\right) = 0$. Hence, $\omega(G) = 0$ which follows that $(x_n)_n$ has a weakly convergent subsequence. Consequently, $\frac{1}{\lambda}A$ is weakly demicompact for all $\lambda \in \mathbb{C}\backslash\Theta$. Now, since $\frac{1}{\lambda}A$ is DP operator, it follows from Corollary 2.1 in [16] that $\lambda I - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\Theta$. Thus, applying Theorem 3.1 in [9], A is a generalized weakly demicompact operator.

Corollary 2.5 Let X be a Banach space, $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$. Suppose there exists an integer $n \geq 1$, such that $(AB)^n$ or $(BA)^n$ is a DP operator and $AB - BA \in \mathcal{W}(X)$. Then, $(AB)^n$ is GWDC if, and only if, $(BA)^n$ is GWDC.

Proof 2.6 *Suppose that $(AB)^n$ is GWDC. In view of Theorem 2.3, there exist a positive constant C_n and a finite subset Θ of $\mathbb{C}$ containing 0 such as for all $G \in \mathcal{M}_X$,*

$$\omega(G) \leq C_n\ \omega\left((\lambda I - (AB)^n)(G)\right), \text{ for all } \lambda \in \mathbb{C}\backslash\Theta.$$

Hence, for every bounded set G,

$$\omega(G) \leq C_n\ \omega\left((\lambda I - (BA)^n)(G)\right) + C_n\ \omega\left(((BA)^n - (AB)^n)(G)\right).$$

Using the following identity

$$(BA)^n - (AB)^n = \sum_{k=0}^{n-1}(BA)^k(BA - AB)(AB)^{n-1-k},$$

we infer that $(BA)^n - (AB)^n \in \mathcal{W}(X)$. Hence, for all $G \in \mathcal{M}_X$,

$$\omega\left(((BA)^n - (AB)^n)(G)\right) = 0.$$

Accordingly, for all $G \in \mathcal{M}_X$,

$$\omega(G) \leq C_n\ \omega\left((\lambda I - (BA)^n)(G)\right), \text{ for all } \lambda \in \mathbb{C}\backslash\Theta.$$

In view of Theorem 2.3, we conclude that $(BA)^n$ is GWDC.
The converse is proved similarly.

Corollary 2.7 Let X be a Banach space and $A \in \mathcal{L}(X)$ such that A is a DP operator. Then, A is GWDC if, and only if, there exists a finite subset Θ of $\mathbb{C}$ containing 0 such that $\omega\left((\lambda I - A)(G)\right) = 0$ implies $\omega(G) = 0$ for all $\lambda \in \mathbb{C}\backslash\Theta$ and for each bounded set $G \subseteq X$.

Proof 2.8 *Suppose that A is generalized weakly demicompact. Then, from Theorem 2.3, there exist a positive constant C and a finite subset Θ of $\mathbb{C}$ containing 0 such that for all $G \in \mathcal{M}_X$,*

$$\omega(G) \leq C \ \ \omega\left((\lambda I - A)(G)\right), \ \textit{for all}\ \lambda \in \mathbb{C}\backslash\Theta.$$

Hence, if $\omega\left((\lambda I - A)(G)\right) = 0$, then $\omega(G) = 0$. Conversely, suppose that there exists a finite subset Θ of $\mathbb{C}$ containing 0 such that $\omega\left((\lambda I - A)(G)\right) = 0$ implies $\omega(G) = 0$ for all $\lambda \in \mathbb{C}\backslash\Theta$ and whenever $G \in \mathcal{M}_X$. Let $(x_n)_n$ be a bounded sequence of X such that

$$(\lambda I - A)x_n \rightharpoonup x \in X.$$

Choose $G = \{x_n; n \in \mathbb{N}\}$, we have so $\omega\left((\lambda I - A)(G)\right) = 0$. Since G is bounded, then $\omega(G) = 0$. Hence, we get $\frac{1}{\lambda}A$ is weakly demicompact for all $\lambda \in \mathbb{C}\backslash\Theta$. Now, from the fact that $\frac{1}{\lambda}A$ is DP operator and applying Corollary 2.1 in [16], we deduce that $\lambda I - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\Theta$. Consequently, we infer from Theorem 3.1 in [9] that A is a generalized weakly demicompact operator. This completes the proof.

Theorem 2.9 Let X be a Banach space and $T \in \mathcal{L}(X)$ such that T is a DP operator. Assume that $\psi_\omega(T^m) < 1$, for some $m > 0$. Then, for every $k \in \mathbb{N}\backslash\{0\}$ and every $\varepsilon \in \{-1, 1\}$, we have εT^k is GWDC.

Proof 2.10 *Let $\lambda \in \mathbb{C}\backslash\{0\}$. Since $\psi_\omega(T^m) < 1$, it follows that $\lim\limits_{n\to+\infty}(\psi_\omega(T^n))^{\frac{1}{n}} = 0$, then there exists $n_0 \geq n$ such that for all $n \geq n_0$,*

$$(\psi_\omega(T^{n_0}))^{\frac{1}{n_0}} < |\lambda|. \tag{1}$$

Also, we can write, for $n \geq n_0$,

$$\lambda^{n_0} - T^{n_0} = \sum_{j=0}^{n_0-1} \lambda^j T^{n_0-1-j}(\lambda - T) = Q(T)(\lambda - T). \tag{2}$$

Let $(x_n)_n$ be a bounded sequence in X such that $y_n := \lambda x_n - Tx_n$ converges weakly to some element $x \in X$. From Equation (2), we have the following inclusion

$$\{\lambda^{n_0}x_n\}_{n\in\mathbb{N}} \subset \{T^{n_0}x_n\}_{n\in\mathbb{N}} + \{Q(T)y_n\}_{n\in\mathbb{N}},$$

then

$$\begin{aligned} |\lambda|^{n_0}\omega\{x_n\}_{n\in\mathbb{N}} &\leq \omega\{T^{n_0}x_n\}_{n\in\mathbb{N}} + \omega\{Q(T)y_n\}_{n\in\mathbb{N}} \\ &\leq \psi_\omega(T^{n_0})\omega\{x_n\}_{n\in\mathbb{N}} + \omega\{Q(T)y_n\}_{n\in\mathbb{N}}. \end{aligned}$$

Since $Q(T)$ is a bounded operator, then it is weakly sequentially continuous, and it follows that $Q(T)y_n \rightharpoonup Q(T)x$. Thus, we get

$$(|\lambda|^{n_0} - \psi_\omega(T^{n_0}))\omega\{x_n\}_{n\in\mathbb{N}} \leq 0.$$

We conclude from Equation (1) that $\omega\{x_n\}_{n\in\mathbb{N}} = 0$. Consequently, in view of Corollary 2.7, T is generalized weakly demicompact. From Proposition 1.1 (iii), we infer that for every $k \in \mathbb{N}\backslash\{0\}$ and every $\varepsilon \in \{-1, 1\}$,

$$\begin{aligned}\psi_\omega(\varepsilon T^k) &= \psi_\omega(T^k)\\ &\leq (\psi_\omega(T))^k\\ &< 1.\end{aligned}$$

Hence, εT^k is a GWDC operator.

3 Left and right essential spectra of the sum of operators

We start this section with the following useful lemma.

Lemma 3.1 [6] Let X be a Banach space and $T, S \in \mathcal{L}(X)$.
(i) If $TS \in \Phi_l(X)$, then $S \in \Phi_l(X)$.
(ii) If $TS \in \Phi_r(X)$, then $T \in \Phi_r(X)$.

Now, we introduce the following set:

$$\Gamma(X) = \{T \in \mathcal{L}(X) : T \text{ is DP and } \psi_\omega(T) < 1\}.$$

Theorem 3.2 Let X be a Banach space, $T \in \mathcal{L}(X)$ and $F \in \mathcal{L}(X)$. Assume that there exists a finite subset Θ of $\mathbb{C}$ containing 0 such that the following assertions hold :
(i) For every $\lambda \in \Phi_{l(T+F)}\backslash\Theta$, there exist $G_{\lambda l}$ and $H_{\lambda l}$ left Fredholm inverses operators of $(\lambda I - T - F)$ such that $\delta TFG_{\lambda l} \in \Gamma(X)$ and $\delta FTH_{\lambda l} \in \Gamma(X)$, for any $\delta \in [0, 1]$. Then

$$[\sigma_l(T) \cup \sigma_l(F)]\backslash\Theta \subset [\sigma_l(T + F)]\backslash\Theta.$$

(ii) For every $\lambda \in \Phi_{r(T+F)}\backslash\Theta$, there exist $G_{\lambda r}$ and $H_{\lambda r}$ right Fredholm inverses operators of $(\lambda I - T - F)$ such that $\delta G_{\lambda r}TF \in \Gamma(X)$ and $\delta H_{\lambda r}FT \in \Gamma(X)$, for any $\delta \in [0, 1]$. Then,

$$[\sigma_r(T) \cup \sigma_r(F)]\backslash\Theta \subset \sigma_r(T + F)\backslash\Theta. \qquad \diamondsuit$$

Proof 3.3 *(i) Let $\lambda \in \mathbb{C}\backslash\Theta$. If there exist $G_{\lambda l}$ and $H_{\lambda l}$ a left Fredholm inverses operators of $(\lambda I - T - F)$ then $G_{\lambda l}(\lambda I - T - F) = I - K$, where $K \in \mathcal{K}(X)$. Then we have*

$$\begin{aligned}(\lambda I - T)(\lambda I - F) &= \lambda(\lambda I - T - F) + TF \\ &= \lambda(\lambda I - T - F) + TFG_{\lambda l}(\lambda I - T - F) + TFK.\end{aligned}$$

So, we obtain

$$(\lambda I - T)(\lambda I - F) = (\lambda I + TFG_{\lambda l})(\lambda I - T - F) + TFK. \tag{3}$$

Also, we can write

$$(\lambda I - F)(\lambda I - T) = (\lambda I + FTH_{\lambda l})(\lambda I - T - F) + FTK. \tag{4}$$

Let $\lambda \notin \sigma_l(T + F) \cup E$, then $\lambda I - T - F \in \Phi_l(X)$ and let $G_{\lambda l}$ and $H_{\lambda l}$ be a left Fredholm inverses operators of $(\lambda I - T - F)$ such that $\delta TFG_{\lambda l} \in \Gamma(X)$ and $\delta FTH_{\lambda l} \in \Gamma(X)$. Hence, from Theorem 2.9, it follows that $-\delta TFG_{\lambda l}$ and $-\delta FTH_{\lambda l}$ are generalized weakly demicompact with a generalized set Θ. Now, when applying Theorem 3.2 in [9], we deduce that $(\lambda I + TFG_{\lambda l})$ and $(\lambda I + FTH_{\lambda l})$ are Fredholm operators on X, for all $\lambda \in \mathbb{C}\backslash\Theta$. Now, taking into account Equations (3) and (4), when applying Theorem 2.5 and Theorem 2.7 in [11], we deduce that $(\lambda I - T)(\lambda I - F) \in \Phi_l(X)$ and $(\lambda I - F)(\lambda I - T) \in \Phi_l(X)$. Thus, using Lemma 3.1, we get $\lambda I - T \in \Phi_l(X)$ and $\lambda I - F \in \Phi_l(X)$, for all $\lambda \in \mathbb{C}\backslash\Theta$.
(ii) If there exist $G_{\lambda r}$ and $H_{\lambda r}$ a right Fredholm inverses operators of $(\lambda I - T - A)$, then we can write

$$(\lambda I - T)(\lambda I - F) = (\lambda I - T - F)(\lambda I + G_{\lambda r}TF) + K_1TF. \tag{5}$$

and

$$(\lambda I - F)(\lambda I - T) = (\lambda I - T - F)(\lambda I + H_{\lambda r}FT) + K_1FT, \tag{6}$$

where $K_1 \in \mathcal{K}(X)$.
The same arguments have been used, it is sufficient to replace Equation (3) by Equation (5) and Equation (4) by Equation (6).

Now, we will extend the results of the above theorem for the left and right Weyl spectra.

Theorem 3.4 Let X be a Banach space and $T, S \in \mathcal{L}(X)$. Assume that there exists a finite subset Θ of $\mathbb{C}$ containing 0.
(i) If for every $\lambda \in \Phi_{l(T+S)}\backslash\Theta$, there exists $G_{\lambda l}$ a left Fredholm inverse of $(\lambda I - T - S)$ such that $\delta TSG_{\lambda l} \in \Gamma(X)$, for any $\delta \in [0, 1]$, then

$$\sigma^l_{ew}(T + S)\backslash\Theta \subset [\sigma^l_{ew}(T) \cup \sigma^l_{ew}(S)]\backslash\Theta.$$

(ii) If for every $\lambda \in \Phi_{r(T+S)} \backslash \Theta$, there exists $G_{\lambda r}$ a right Fredholm inverse of $(\lambda I - T - S)$ such that $\delta G_{\lambda r} TS \in \Gamma(X)$, for any $\delta \in [0,1]$, then

$$\sigma^r_{ew}(T+S) \backslash \Theta \subset [\sigma^r_{ew}(T) \cup \sigma^r_{ew}(S)] \backslash \Theta.$$

Proof 3.5 *(i) Let $\lambda \in \mathbb{C} \backslash \Theta$. If there exists $G_{\lambda l}$ a left Fredholm inverse operator of $(\lambda I - T - S)$ then $G_{\lambda l}(\lambda I - T - S) = I - K$, where $K \in \mathcal{K}(X)$. Then we have*

$$\begin{aligned} (\lambda I - T)(\lambda I - S) &= \lambda(\lambda I - T - S) + TS \\ &= \lambda(\lambda I - T - S) + TSG_{\lambda l}(\lambda I - T - S) + TSK. \end{aligned}$$

So, we obtain

$$(\lambda I - T)(\lambda I - S) = (\lambda I + TSG_{\lambda l})(\lambda I - T - S) + TSK. \quad (7)$$

Let $\lambda \notin [\sigma^l_{ew}(T) \cup \sigma^l_{ew}(S)] \cup E$, then

$$\lambda I - T \in \Phi_l(X) \text{ and } i(\lambda I - T) \leq 0,$$

$$\lambda I - S \in \Phi_l(X) \text{ and } i(\lambda I - S) \leq 0.$$

Thus, the use of Theorem 2.5 and Theorem 2.7 in [11] and Equation (7), enables us to deduce that

$$(\lambda I + TSG_{\lambda l})(\lambda I - T - S) \in \Phi_l(X) \text{ and } i((\lambda I + TSG_{\lambda l})(\lambda I - T - S)) \leq 0.$$

As $\delta TSG_{\lambda l} \in \Gamma(X)$, we infer from Theorem 2.9 that, $-\delta TSG_{\lambda l}$ is GWDC with a generalized set Θ. Now, when applying Theorem 3.2 in [9], we get

$$\lambda I + TSG_{\lambda l} \in \Phi(X) \text{ and } i(\lambda I + TSG_{\lambda l}) = 0.$$

Hence, from Lemma 3.1, we deduce that

$$\lambda I - T - S \in \Phi_l(X) \text{ and } i(\lambda I - T - S) \leq 0.$$

Hence, $\lambda \notin \sigma^l_{ew}(T+S)$.
(ii) This assertion can be proved similarly to the first item.

4 Open Problems

1. Can we relax the DP property in Theorems 2.3 and 2.9 ?
2. Can we extend these results for unbounded linear operators with an axiomatic measure of weak noncompactness?
3. What about generalized essential spectra of the sum of two bounded or unbounded linear operators (see [3])?

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