

# Connected Triangular Norms and Applications

Mustapha Raïssouli, and Mohamed Chergui

Department of Mathematics, Science Faculty, Moulay Ismail University, Meknes  
Morocco

e-mail:raïssouli.mustapha@gmail.com

Department of Mathematics, CRMEF-RSK, EREAM Team, LaREAMI-Lab, Kenitra  
Morocco

e-mail:chergui\_m@yahoo.fr

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## Abstract

*The fundamental goal of this paper is to introduce some new concepts for triangular norms. As application, we provide a classification of the standard triangular norms known in the literature as well as some simple relationships between them. Our approach generates a lot of open problems as purpose for future research.*

**Keywords:** *t-norm, smooth t-norm, connected t-norms.*

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## 1 Introduction

Recently, considerable attention has been paid to the theory of triangular norms introduced by Menge [7] and widely studied by Schwetzer and Sklar [10] in the context of statistical metric spaces. It represents an extension of the classical logical operator "AND" to the real of fuzzy logic. In particular, it serves efficiently in solving problems of multicriteria decision making [11].

Let us begin by stating the axiomatic definition of triangular norms. A map  $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$  is called triangular-norm, briefly *t-norm*, if the following requirements are satisfied.

- (i)  $T(x, y) = T(y, x)$  for all  $x, y \in [0, 1]$ , ( $T$  is commutative)
- (ii)  $T(x_1, y_1) \leq T(x_2, y_2)$  whenever  $x_1 \leq x_2$  and  $y_1 \leq y_2$ , ( $T$  is increasing)

- (iii)  $T(x, T(y, z)) = T(T(x, y), z)$  for all  $x, y, z \in [0, 1]$ , ( $T$  is associative)
- (iv)  $T(x, 1) = x$  for all  $x \in [0, 1]$ , ( $T$  has 1 as identity).

As it is well known, combining (i), (ii) and (iv) we can see that every  $t$ -norm  $T$  satisfies  $T(x, y) \leq \min(x, y)$  for all  $x, y \in [0, 1]$ . It follows that  $T(x, x) \leq x$  for all  $x \in [0, 1]$ , with  $T(0, 0) = 0$  and  $T(1, 1) = 1$ . Continuity and differentiability of a  $t$ -norm  $T$  are defined, as usual, when considering  $T$  as a function in two variables  $x, y \in [0, 1]$ . We say that  $T$  is Archimedean if  $T$  is continuous and satisfies  $T(x, x) < x$  for all  $x \in (0, 1)$ . Note that the condition  $T(x, x) < x$  for all  $x \in (0, 1)$  is equivalent to say that the map  $x \mapsto T(x, x)$  has no fixed point on  $(0, 1)$ .

The standard examples of  $t$ -norms are stated in the following, see [1, 2, 4, 5, 8, 9]:

$$M(x, y) = \min(x, y); \quad \Pi(x, y) = xy; \quad L(x, y) = \max(x + y - 1, 0);$$

$$W(x, 1) = x, \quad W(1, y) = y, \quad W(x, y) = 0 \text{ if } x, y \in [0, 1];$$

$$N(x, y) = \min(x, y) \text{ if } x + y \geq 1, \quad N(x, y) = 0 \text{ if } x + y < 1;$$

$$H(x, y) = \frac{xy}{x + y - xy}, \text{ with } H(0, 0) = 0;$$

and are known in the literature as the minimum  $t$ -norm, product  $t$ -norm, Lukasiewicz  $t$ -norm, drastic or weak  $t$ -norm, nilpotent  $t$ -norm and Hamacher  $t$ -norm, respectively. The  $t$ -norms  $W$  and  $N$  are obviously discontinuous. It is easy to see that  $M$  is continuous but not Archimedean and not differentiable at  $(a, a)$ , for any  $a \in (0, 1)$ ,  $\Pi$  is Archimedean and differentiable,  $H$  is Archimedean but not differentiable at  $(0, 0)$  and  $L$  is Archimedean not differentiable at  $(a, b) \in [0, 1] \times [0, 1]$  such that  $a + b = 1$ .

In the literature, we can find a lot of families of  $t$ -norms which are extensions of some of the previous standard  $t$ -norms. For instance, we cite [2]

$$H^{(\gamma)}(x, y) = \frac{xy}{\gamma + (1 - \gamma)(x + y - xy)}, \quad \gamma \geq 0;$$

$$D^{(\alpha)}(x, y) = \frac{xy}{\max(x, y, \alpha)}, \quad \alpha \in (0, 1),$$

known as the (parameterized) Hamacher  $t$ -norm and Dubois-Prade  $t$ -norm, respectively. Clearly, we have

$$H^{(0)} = H, \quad H^{(1)} = \Pi \text{ and } D^{(0)} := \lim_{\alpha \downarrow 0} D^{(\alpha)} = W, \quad D^{(1)} := \lim_{\alpha \uparrow 1} D^{(\alpha)} = \Pi.$$

It is easy to check that  $H^{(\gamma)}$  is Archimedean and differentiable for each  $\gamma > 0$  whereas  $D^{(\alpha)}$  is continuous but neither Archimedean nor differentiable, for  $\alpha \in (0, 1)$ . For other families of  $t$ -norms (not needed here) we refer the interested reader to [2] for instance.

**Remark 1.1.** (i) We precise that the parameters  $\gamma \geq 0$  and  $\alpha \in (0, 1)$  in the  $t$ -norms  $H^{(\gamma)}$  and  $D^{(\alpha)}$  were written with superscript in the aim to avoid any confusion with other needed notations used later.

(ii) The set of all  $t$ -norms can be equipped with the point-wise partial order defined as follows: For  $t$  norms  $T_1$  and  $T_2$ ,  $T_1 \leq T_2$  means that  $T_1(a, b) \leq T_2(a, b)$  for all  $a, b \in [0, 1]$ . In this case, we say that  $T_2$  is stronger than  $T_1$ .

Otherwise, let  $\Phi : [0, 1] \rightarrow [0, 1]$  be a continuous strictly increasing map which satisfies  $\Phi(0) = 0$  and  $\Phi(1) = 1$ . Such map will be called a connection throughout the following. Clearly, if  $\Phi$  is a connection then so is  $\Phi^{-1}$ , inverse of  $\Phi$ . The simplest connection is the identity map  $id : x \mapsto x$  which will be called the trivial connection. Also, the map  $x \mapsto x^p$  is a connection for each real number  $p > 0$ . Clearly,  $x \mapsto 1 - x$  is not a connection.

Let  $\Phi$  be a connection. If for all  $x, y \in [0, 1]$ , we set

$$T_{\Phi}(x, y) = \Phi^{-1}\left(T(\Phi(x), \Phi(y))\right),$$

then it is well-known that  $T_{\Phi}$  is a  $t$ -norm. Obviously  $T_{id} = T$ . It is easy to see that if  $T$  is Archimedean then it is the same for  $T_{\Phi}$  for every connection  $\Phi$ . The following assertions are satisfied:

- (a)  $(T_{\Phi})_{\Psi} = T_{\Phi \circ \Psi}$  and so  $(T_{\Phi})_{\Phi^{-1}} = T$ , for all connections  $\Phi$  and  $\Psi$ .
- (b) If  $T$  and  $S$  are two  $t$ -norms such that  $T_{\Phi} = S_{\Phi}$  for some connection  $\Phi$ , then  $T = S$ .
- (c) If  $T$  and  $S$  are two  $t$ -norms such that  $T \leq S$  then  $T_{\Phi} \leq S_{\Phi}$  for all connection  $\Phi$ .
- (d)  $T_{\Phi} = T_{\Psi}$  does not ensure  $\Phi = \Psi$ .

**Example 1.1.** Let  $\Phi_p$  be the connection defined by  $\Phi_p(x) = x^p$ ,  $p > 0$ . We obtain, the power  $t$ -norm  $T_p$  defined by

$$\forall x, y \in [0, 1] \quad T_p(x, y) =: T_{\Phi_p}(x, y) = \left(T(x^p, y^p)\right)^{1/p}.$$

For a detailed study of  $T_p$  as well as its application for the previous standard  $t$ -norms, see [3].

**Remark 1.2.** Let  $(\Phi_n)$  be a sequence of connections. If  $(\Phi_n)$  converges point-wisely to  $\Phi$ , i.e.  $(\Phi_n(x))$  converges (in  $\mathbb{R}$ ) to  $\Phi(x)$  for all  $x \in [0, 1]$ , the function  $\Phi$  is not necessary a connection. Indeed,  $\Phi_n(x) = x^n$  is a connection for each  $n \geq 1$ , its point-wise limit  $\Phi$  is  $\Phi(x) = 1$  if  $x = 1$  and  $\Phi(x) = 0$  if  $x \in [0, 1)$ , with  $\Phi$  is not a connection (since  $\Phi$  is neither continuous nor strictly increasing).

## 2 Smooth $t$ -norm, self-connected $t$ -norm

In this section, we define new concepts related to  $t$ -norms. Namely, smoothness and self-connection properties will be stated.

Let  $T$  be a  $t$ -norm. For all  $x \in [0, 1]$ , we set  $g_T(x) = T(x, x)$ . We then define an increasing function  $g_T : [0, 1] \rightarrow [0, 1]$  which will be called the generating function of  $T$ . It is clear that  $g_T(x) \leq x$  for all  $x \in [0, 1]$ , with  $g_T(0) = 0$  and  $g_T(1) = 1$ . The following lemmas will be needed in the sequel.

**Lemma 2.1.** *Let  $\Phi$  be a connection. Then the following relationship holds*

$$g_{T_\Phi} = \Phi^{-1} \circ g_T \circ \Phi. \quad (1)$$

*Proof.* For all  $x \in [0, 1]$  we have

$$g_{T_\Phi}(x) = T_\Phi(x, x) = \Phi^{-1}\left(T(\Phi(x), \Phi(x))\right) = \Phi^{-1}\left(g_T(\Phi(x))\right) = \Phi^{-1} \circ g_T \circ \Phi(x),$$

whence (1).  $\square$

**Lemma 2.2.** *Let  $T$  be a  $t$ -norm. Then the following equality*

$$g_T\left(T(x, y)\right) = T\left(g_T(x), g_T(y)\right)$$

*holds for all  $x, y \in [0, 1]$ .*

*Proof.* By virtue of the commutativity and associativity of  $T$ , the following relation

$$T\left(T(x, y), T(x, y)\right) = T\left(T(x, x), T(y, y)\right)$$

holds true for all  $x, y \in [0, 1]$ . The desired result follows by the definition of  $g_T$ .  $\square$

We now state the following definition.

**Definition 2.3.** *Let  $T$  be a  $t$ -norm. We say that  $T$  is smooth if the map  $x \mapsto g_T(x) := T(x, x)$  is a connection. We also say that  $T$  generates a connection and  $g_T$  is the generating connection of  $T$ .*

**Proposition 2.1.** *If  $T$  is a smooth  $t$ -norm then so is  $T_\Phi$  for any connection  $\Phi$ .*

*Proof.* If  $T$  is smooth we deduce from (1) that  $g_{T_\Phi}$  is a connection, as composite of three connections.  $\square$

The following proposition deals with the smoothness property of the previous standard  $t$ -norms.

**Proposition 2.2.** *The following assertions are true:*

- (i) *The  $t$ -norms  $M, \Pi$  and  $H$  are smooth whereas  $W, L$  and  $N$  are not.*
- (ii) *The parameterized Hamacher  $t$ -norm  $H^{(\gamma)}$  is smooth for any  $\gamma > 0$ .*
- (iii) *The Dubois-Prade  $t$ -norm  $D^{(\alpha)}$  is also smooth for every  $\alpha \in (0, 1)$ .*

*Proof.* It is a simple exercise of Real Analysis. The details are straightforward and therefore omitted here for the reader.  $\square$

We now state the following definition.

**Definition 2.4.** *Let  $T$  be a  $t$ -norm. We say that  $T$  is:*

- (i) *totally self-connected if  $T_{\Phi} = T$  for any connection  $\Phi$ .*
- (ii) *self-connected if  $T_{\Phi} = T$  for a nontrivial connection  $\Phi$ . We then say that  $T$  is  $\Phi$ -self-connected. If  $T$  is  $\Phi$ -self-connected with  $\Phi(x) = x^p$  for some  $p > 0, p \neq 1$ , we then say that  $T$  is  $p$ -self-connected.*
- (iii) *A  $t$ -norm  $T$  for which there is no trivial connection  $\Phi$  such that  $T_{\Phi} = T$  will be called self-disconnected.*

It is obvious that if  $T$  is  $\Phi$ -self-connected then  $T$  is also  $\Phi^{-1}$ -self-connected. In particular, if  $T$  is  $p$ -self-connected then  $T$  is  $1/p$ -self-connected. Moreover, if  $T$  is  $\Phi$ -self-connected then  $T$  is  $\Phi^n$ -self-connected for all integer  $n \geq 1$ , where  $\Phi^n$  denotes the  $n$ -iterate of  $\Phi$  i.e.  $\Phi^n = \Phi \circ \Phi \circ \dots \circ \Phi$ . So, if  $T$  is  $p$ -self-connected then  $T$  is  $p^n$ -self-connected for every integer  $n$ .

**Example 2.5.** (i) *We can see that  $M_{\Phi} = M$  and  $W_{\Phi} = W$  for all connection  $\Phi$ . Consequently, the  $t$ -norms  $M$  and  $W$  are both totally self-connected.*  
(ii) *It is easy to verify that  $\Pi$  is  $p$ -self-connected for every real number  $p > 0, p \neq 1$ .*

The following result gives further examples of interest.

**Proposition 2.3.** *Let  $T$  be a smooth  $t$ -norm with its generating connection  $g_T$ . Then  $T$  is  $g_T$ -self-connected.*

*Proof.* By Lemma 2.2, we have, for all  $x, y \in [0, 1]$ ,

$$g_T(T(x, y)) = T(g_T(x), g_T(y))$$

or equivalently

$$T(x, y) = g_T^{-1}(T(g_T(x), g_T(y))) = T_{g_T}(x, y).$$

The proposition is so proved.  $\square$

**Corollary 2.6.** (i) *The  $t$ -norms  $M, \Pi$  and  $H$  are self-connected.*  
(ii) *For any  $\gamma \geq 0, H^{(\gamma)}$  is self-connected.*  
(iii) *For every  $\alpha \in (0, 1), D^{(\alpha)}$  is self-connected.*

*Proof.* It is sufficient to combine Proposition 2.2 with Proposition 2.3.  $\square$

### 3 Connection property for $t$ -norms

In the present section, we continue defining new concepts for  $t$ -norms and determining some of their properties.

#### 3.1 Connected $t$ -norms

The current subsection is focused on a special class of  $t$ -norms. For this, let us begin by stating the following definition.

**Definition 3.1.** *Let  $T$  and  $S$  be two  $t$ -norms. We say that  $T$  is connected to  $S$  if  $T_\Phi = S$  for some connection  $\Phi$ .*

It is clear that if  $T$  is connected to  $S$  then  $S$  is connected to  $T$ , since  $T_\Phi = S$  implies  $S_{\Phi^{-1}} = T$ . We then say that  $T$  and  $S$  are connected. If we define

$$T \equiv S \text{ if and only if } T \text{ and } S \text{ are connected,}$$

then  $\equiv$  is obviously an equivalence relation on the set of  $t$ -norms. The class of equivalence of  $T$ , next called network of  $T$  and denoted by  $net(T)$  is then  $net(T) = \{T_\Phi, \Phi \text{ connection}\}$ . With this, it is clear that  $T$  is totally self-connected if and only if its network is a singleton i.e.  $net(T) = \{T\}$ . If  $T$  and  $S$  are connected then they have the same network. We say that  $T$  and  $S$  are disconnected if they are not connected, that is there is no connection  $\Phi$  for which  $T_\Phi = S$ . Clearly,  $T$  and  $S$  are disconnected if and only if  $net(T) \cap net(S) = \emptyset$ .

The following result may be stated.

**Proposition 3.1.** *Let  $T$  and  $S$  be two connected  $t$ -norms. Then we have:*

- (i)  $T$  and  $S$  are both continuous or both discontinuous.
- (ii)  $T$  and  $S$  are both smooth or both non-smooth.
- (iii)  $\Phi \circ g_T = g_S \circ \Phi$  provided  $T_\Phi = S$  for some connection  $\Phi$ .

*Proof.* Follows from simple manipulations and arguments of Real Analysis. The details are omitted here.  $\square$

We now state a list of results which gives interesting examples of connected (resp. disconnected)  $t$ -norms.

**Proposition 3.2.** *Let  $\gamma > 0$  and let  $\Phi_\gamma$  be defined as follows*

$$\forall x \in [0, 1] \quad \Phi_\gamma(x) = \frac{x}{\gamma + (1 - \gamma)x}.$$

*Then we have*

$$\Pi_{\Phi_\gamma} = H^{(\gamma)}.$$

*That is,  $\Pi$  and  $H^{(\gamma)}$  are connected, for any  $\gamma > 0$ .*

*Proof.* An elementary computation leads to, for all  $x, y \in [0, 1]$ ,

$$\Phi_\gamma(x)\Phi_\gamma(y) = \frac{xy}{(\gamma + (1 - \gamma)x)(\gamma + (1 - \gamma)y)},$$

$$\Phi_\gamma(H^{(\gamma)}(x, y)) = \frac{xy}{\gamma^2 + (1 - \gamma)xy + \gamma(1 - \gamma)(x + y - xy)}.$$

A simple observation for comparing the two previous expressions ensures that they are equal and so the desired result is obtained.  $\square$

**Proposition 3.3.** *Let  $\alpha \in (0, 1)$ . Then  $\Pi$  and  $D^{(\alpha)}$  are disconnected.*

*Proof.* Assume that  $\Pi_\Phi = D^{(\alpha)}$  for some connection  $\Phi$  i.e. for all  $x, y \in [0, 1]$  we have

$$\Phi(x)\Phi(y) = \Phi\left(\frac{xy}{\max(x, y, \alpha)}\right).$$

In particular, if  $y = x$  we should have, for all  $x \in [0, 1]$ ,

$$(\Phi(x))^2 = \Phi\left(\frac{x^2}{\max(x, \alpha)}\right).$$

If  $x \in (\alpha, 1)$  we easily deduce that  $(\Phi(x))^2 = \Phi(x)$ . Since  $\Phi$  is a connection then  $\Phi(x) = 0$  or  $\Phi(x) = 1$  that is,  $x = 0$  or  $x = 1$ . This contradicts  $x \in (\alpha, 1)$ , and the desired result is proved.  $\square$

**Corollary 3.2.** *For every  $\gamma > 0$  and  $\alpha \in (0, 1)$ ,  $H^{(\gamma)}$  and  $D^{(\alpha)}$  are disconnected.*

*Proof.* Following Proposition 3.2 we have  $\text{net}(\Pi) = \text{net}(H^{(\gamma)})$  for every  $\gamma > 0$ . By Proposition 3.3 one has  $\text{net}(\Pi) \cap \text{net}(D^{(\alpha)}) = \emptyset$  for each  $\alpha \in (0, 1)$ . We then deduce that  $\text{net}(H^{(\gamma)}) \cap \text{net}(D^{(\alpha)}) = \emptyset$  for every  $\gamma > 0$  and  $\alpha \in (0, 1)$ , and the desired result follows.  $\square$

**Proposition 3.4.** *Let  $\alpha \in (0, 1)$ . Then  $H$  and  $D^{(\alpha)}$  are disconnected.*

*Proof.* It is very similar to that of Proposition 3.3. The details are simple and therefore omitted here for the reader as an interesting exercise.  $\square$

We now give the following result.

**Proposition 3.5.** *The two  $t$ -norms  $L$  and  $N$  are disconnected.*

*Proof.* We can prove this result in two different ways:

(a) Assume that  $L = N_\Phi$  for a connection  $\Phi$ . By Proposition 3.1,(iii) we should have

$$\forall x \in [0, 1] \quad \Phi \circ g_L(x) = g_N \circ \Phi(x).$$

Remark that  $g_L(x) = 2x - 1$  if  $x \geq 1/2$  and  $g_L(x) = 0$  else,  $g_N(x) = x$  if  $x \geq 1/2$  and  $g_N(x) = 0$  if not. Now, let  $x_0$  be such that  $\Phi(x_0) = 1/2$ . Then we must have

$$\Phi(g_L(x_0)) = g_N(\Phi(x_0)) = \Phi(x_0).$$

This, with the fact that  $\Phi$  is one-to-one, yields  $g_L(x_0) = x_0$ . Since  $L$  is Archimedean we then deduce that  $x_0 = 0$  or  $x_0 = 1$ . This is in contradiction with  $\Phi(x_0) = 1/2$ . The desired result is so obtained.

(b) It is clear that  $L$  is continuous but  $N$  is not. By Proposition 3.1.(i) we immediately deduce that  $L$  and  $N$  can not be connected.  $\square$

### 3.2 Weak connected $t$ -norms, strong connected $t$ -norms

In this subsection, we introduce another concept between  $t$ -norms. Precisely, we state the following.

**Definition 3.3.** *Let  $T$  and  $S$  be two  $t$ -norms. We say that:*

(i)  *$T$  is weakly connected to  $S$  (or  $T$  is  $S$ -weakly connected) if there exist a sequence  $(\Phi_n)$  of connections such that  $(S_{\Phi_n})_n$  converges point-wisely to  $T$ , i.e.*

$$\forall x, y \in [0, 1] \quad \lim_{n \uparrow \infty} S_{\Phi_n}(x, y) = T(x, y).$$

(ii)  *$T$  is strongly connected to  $S$  (or  $T$  is  $S$ -strongly connected) if there exist a sequence  $(\Phi_n)$  of connections such that  $(S_{\Phi_n})_n$  converges uniformly on  $[0, 1] \times [0, 1]$  to  $T$ , i.e.*

$$\lim_{n \uparrow +\infty} \sup_{x, y \in [0, 1]} |S_{\Phi_n}(x, y) - T(x, y)| = 0.$$

**Proposition 3.6.** *Let  $(\Phi_n)$  be the sequence of connections defined by*

$$\forall x \in [0, 1] \quad \Phi_n(x) = \frac{2^n x}{(2^n - 1)x + 1}.$$

*Then the following assertions hold:*

(i) *For all  $x, y \in [0, 1]$  we have*

$$\Pi_{\Phi_n}(x, y) = \frac{2^n xy}{(2^n - 1)(x + y - xy) + 1} = H^{(1/2^n)}(x, y).$$

(ii) *For all  $x, y \in [0, 1]$  one has*

$$\lim_{n \uparrow \infty} \Pi_{\Phi_n}(x, y) = \frac{xy}{x + y - xy} := H(x, y).$$

*That is,  $H$  is  $\Pi$ -weakly connected.*

*Proof.* (i) Follows from Proposition 3.2 when we take  $\gamma = 1/n$  for fixed  $n \geq 1$ .  
(ii) From (i), we can write

$$\Pi_{\Phi_n}(x, y) = \frac{xy}{(1 - 2^{-n})(x + y - xy) + 2^{-n}},$$

from which the desired result follows.  $\square$

We end this section by stating the following remark which may be of interest.

**Remark 3.7.** *It is worth mentioning that the previous sequence of  $t$ -norms  $\Pi_{\Phi_n}$  is differentiable on  $[0, 1] \times [0, 1]$  (for each  $n \geq 0$ ) and approximates the non-differentiable  $t$ -norm  $H$ , since  $H$  is not differentiable at  $(0, 0)$ .*

## 4 Conclusion and open problems

In this section, we provide a summarize of the previous defined concepts and their related results. Afterwards, we will resort to a list of open problems as purposes for future work.

$t$ -norm	Smoothness property	Self-connection property
$M$	Yes	Yes
$\Pi$	Yes	Yes
$H$	Yes	Yes
$W$	No	Yes
$L$	No	Open problem
$N$	No	Open problem
$H^{(\gamma)} \gamma > 0$	Yes	Yes
$D^{(\alpha)} \alpha \in (0, 1)$	Yes	Yes

Table 1: Summary about smooth and self-connected  $t$ -norms

TABLE 1 presented an overview on some results previously stated. The following is an open problem.

**Problem 1:** *Prove or disprove that  $L$  and  $N$  are self-connected.*

$t$ -norm	$H^{(\gamma)}$	$D^{(\alpha)}$	$N$
$\Pi$	connected	disconnected	disconnected
$H$	Open problem	disconnected	disconnected
$L$	disconnected	disconnected	disconnected
$D^{(\alpha)}$	disconnected	connected	disconnected

Table 2: Connected and disconnected  $t$ -norms.

In TABLE 2, we used Proposition 3.1, namely: if two  $t$ -norms are connected then they are both continuous or both discontinuous (resp. both smooth or both non-smooth). The following problem is open.

**Problem 2:** Let  $\gamma \in (0, 1)$ . Prove or disprove that  $H^{(\gamma)}$  and  $H$  are connected.

Finally, we state the following.

**Problem 3:** How can the study provided in this work be generalized by considering the concept of  $t$ -conorm (or dual) in the sense defined in [6]? In particular, is-it possible to classify more  $t$ -norms by considering their duals?

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