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On a conjecture for subclasses of analytic and p-valent functions

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Abstract

Let $\mathcal{A}_p$ be the class of $f(z)$ which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Two subclasses $S_p^*(\alpha)$ and $K_p(\alpha)$ of $\mathcal{A}_p$ are considered. For the case of $p = 1$, MacGregor derives an interesting result for subordinations of $f(z) \in \mathcal{A}_1$. With the result by MacGregor, Wilken and Feng show that if $f(z) \in K_1(\alpha)$ then $f(z) \in S_1^*(\beta)$ and β is given. The object of the present paper is to introduce one conjecture between $S_p^*(\alpha)$ and $K_p(\alpha)$ concerning with the results by MacGregor, Wilken and Feng.

Keywords: *Analytic p-valent function, Starlike function of order α, Convex function of order α, Subordination.*

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1 Introduction

Let $\mathcal{A}_p$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad , \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\} \tag{1}$$

that are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}_p$, we say that $f(z) \in S_p^*(\alpha)$ if $f(z)$ satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{U} \tag{2}$$

for some $\alpha (0 \leq \alpha < p)$. Also, we introduce the subclass $K_p(\alpha)$ of $\mathcal{A}_p$ consisting of $f(z)$ that satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,,\, z \in \mathbb{U} \tag{3}$$

for some $\alpha (0 \leq \alpha < p)$.

Let $f(z) \in \mathcal{A}_p$ and $g(z) \in \mathcal{A}_p$. Then, $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. Miller and Mocanu [2], Pommerenke [3]).

For $f(z) \in \mathcal{A}_1$ $(p = 1)$, MacGregor [1] proves that if $f(z) \in K_1(\alpha)$, then

$$\frac{zf'(z)}{f(z)} \prec g(z), \quad z \in \mathbb{U} \tag{4}$$

where

$$g(z) = \begin{cases} (2\alpha - 1)\frac{z}{(1-z)^{2(1-\alpha)}(1-(1-z)^{2\alpha-1})} & , \ (\alpha \neq \frac{1}{2}) \\ \\ -\frac{z}{(1-z)\log(1-z)} & , \ (\alpha = \frac{1}{2}). \end{cases} \tag{5}$$

Concerning with the subordination (4), Wilken and Feng [4] derive the following result: If $f(z) \in K_1(\alpha)$, then $f(z) \in S_1^*(\beta)$, where

$$\beta = g(-1) = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})} & , \ (\alpha \neq \frac{1}{2}) \\ \\ \frac{1}{2\log 2} & , \ (\alpha = \frac{1}{2}). \end{cases} \tag{6}$$

With the above fact, we consider one conjecture between $S_p^*(\alpha)$ and $K_p(\alpha)$.

2 Conjecture

By means of facts by MacGregor [1], and by Wilken and Feng [4], we can guess the following conjecture.

Conjecture: If $f(z) \in K_p(\alpha)$, then

$$\frac{zf'(z)}{f(z)} \prec g(z), \quad z \in \mathbb{U} \tag{7}$$

where

$$g(z) = \begin{cases} \frac{z^p}{(1-z)^{2(p-\alpha)} \int_0^z \frac{t^{p-1}}{(1-t)^{2(p-\alpha)}} dt} & , (\alpha \neq \frac{p}{2}) \\ \\ \frac{z^p}{(1-z)^p \int_0^z \frac{t^{p-1}}{(1-t)^p} dt} & , (\alpha = \frac{p}{2}). \end{cases} \tag{8}$$

If $f(z) \in K_p(\alpha)$ then $f(z) \in S_p^*(\beta)$, where $\beta = g(-1)$.

Example 2.1 *The function $g(z)$ of (8) is given by two integrations*

$$I_1 = \int_0^z \frac{t^{p-1}}{(1-t)^{2(p-\alpha)}} dt \tag{9}$$

$$I_2 = \int_0^z \frac{t^{p-1}}{(1-t)^p} dt. \tag{10}$$

We consider the change of variable $1 - t = u$. Then

$$I_1 = \int_{1-z}^1 (1-u)^{p-1} u^{-2(p-\alpha)} du \tag{11}$$

$$I_2 = \int_{1-z}^1 (1-u)^{p-1} u^{-p} du. \tag{12}$$

Note that

$$(1-u)^{p-1} = 1 + \sum_{j=1}^{p-1} \frac{(p-1)(p-2)\cdots(p-j)}{j!} (-u)^j. \tag{13}$$

Therefore, using (13), we have

$$I_1 = \int_{1-z}^1 u^{2\alpha-2p} \left(1 + \sum_{j=1}^{p-1} \frac{(p-1)(p-2)\cdots(p-j)}{j!} (-u)^j \right) du \tag{14}$$

and

$$I_2 = \int_{1-z}^1 u^{-p} \left(1 + \sum_{j=1}^{p-1} \frac{(p-1)(p-2)\cdots(p-j)}{j!} (-u)^j \right) du. \tag{15}$$

With this fact, we know that we obtain the function $g(z)$ in (8).

We consider the case of $p = 2$ in (8). Then $g(z)$ is given by

$$g(z) = \begin{cases} \frac{z^2}{(1-z)^{2(2-\alpha)}\int_0^z \frac{t}{(1-t)^{2(2-\alpha)}}dt} & , (\alpha \neq 1) \\ \\ \frac{z^2}{(1-z)^2\int_0^z \frac{t}{(1-t)^2}dt} & , (\alpha = 1). \end{cases} \tag{16}$$

where $0 \leq \alpha < 2$. Using the change of variable $1 - t = u$, we have

$$\begin{aligned} \int_0^z \frac{t}{(1-t)^{2(2-\alpha)}}dt &= \int_{1-z}^1 (u^{2\alpha-4} - u^{2\alpha-3})du \\ &= \frac{1}{2(\alpha-1)(2\alpha-3)}\left\{1 - (1-z)^{2\alpha-3} - (2\alpha-3)z(1-z)^{2\alpha-3}\right\} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \int_0^z \frac{t}{(1-t)^2}dt &= \int_{1-z}^1 (u^{-2} - u^{-1})du \\ &= \frac{z}{1-z} + log(1-z). \end{aligned} \tag{18}$$

Therefore, we say that

$$g(z) = \begin{cases} \frac{2(\alpha-1)(2\alpha-3)z^2}{(1-z)^{2(2-\alpha)}\{1-(1-z)^{2\alpha-3}-(2\alpha-3)z(1-z)^{2\alpha-3}\}} & , (\alpha \neq 1) \\ \\ \frac{z^2}{(1-z)(z+(1-z)log(1-z))} & , (\alpha = 1). \end{cases} \tag{19}$$

This $g(z)$ shows us that

$$g(-1) = \begin{cases} \frac{2(\alpha-1)(2\alpha-3)}{2^{2(2-\alpha)}\{1-2^{2\alpha-3}+(2\alpha-3)2^{2\alpha-3}\}} & , (\alpha \neq 1) \\ \\ \frac{-1}{2(1-2log2)} = 1.25647\ldots & , (\alpha = 1). \end{cases} \tag{20}$$

If we take $\alpha = \frac{1}{2}$ in (20), then we have $g(-1) = 1$.

3 Appendix

Let us consider $f(z) \in \mathcal{A}_1$ for $p = 1$. Then, the result by MacGregor [1], and by Wilken and Feng [4] give that if $f(z) \in \mathcal{A}_1$ satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,,\, z \in \mathbb{U} \tag{21}$$

for some $\alpha(0 \leq \alpha < 1)$, then

$$\frac{zf'(z)}{f(z)} \prec g(z),\ z \in \mathbb{U} \tag{22}$$

for given by (5), and

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta = g(-1)\ ,\ z \in \mathbb{U}. \tag{23}$$

Now, we consider a function $F(z)$ defined by

$$F(z) = \frac{zf'(z)}{f(z)},\ (f(z) \in \mathcal{A}_1). \tag{24}$$

Then we say that if $F(z)$ satisfies

$$Re\left\{\frac{zF'(z)}{F(z)} + F(z)\right\} = Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ z \in \mathbb{U} \tag{25}$$

for some real $\alpha(0 \leq \alpha < 1)$, then

$$F(z) = \frac{zf'(z)}{f(z)} \prec g(z),\ z \in \mathbb{U} \tag{26}$$

and

$$ReF(z) = Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta = g(-1),\ z \in \mathbb{U}. \tag{27}$$

Since $F(z)$ is analytic in $\mathbb{U}$ and $F(0) = 1$, we take

$$F(z) = \frac{f(z)}{z},\ \ (f(z) \in \mathcal{A}_1). \tag{28}$$

Then we have that if $f(z) \in \mathcal{A}_1$ satisfies

$$Re\left\{\frac{zF'(z)}{F(z)} + F(z)\right\} = Re\left\{\frac{zf'(z)}{f(z)} + \frac{f(z)}{z} - 1\right\} > \alpha,\ z \in \mathbb{U} \tag{29}$$

for some real $\alpha(0 \leq \alpha < 1)$, then

$$F(z) = \frac{f(z)}{z} \prec g(z),\ z \in \mathbb{U} \tag{30}$$

and

$$ReF(z) = Re\left\{\frac{f(z)}{z}\right\} > \beta = g(-1),\ z \in \mathbb{U}. \tag{31}$$

4 Open Problem

We note that if we take $p = 1$ in our conjecture, then $g(z)$ of (8) becomes $g(z)$ of (5), and $S_p^*(\beta)$ in our conjecture becomes $S_1^*(\beta)$ of (6). Therefore, we guess our conjecture for $S_p^*(\alpha)$ and $K_p(\alpha)$ is true. We hope our conjecture will be prove near future.

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