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Amenability and Connes amenability related to generalized module extension Banach algebras

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Abstract

We study amenability and preduality of generalized module extension Banach algebras. Besides, we shall consider issues of Connes-amenability of these algebras and of their second dual Banach spaces endowed with the corresponding Arens products.

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1 Introduction

In this article we shall analyze the issue of amenability and Connes-amenability of generalized module extension Banach algebras. To be precise, let A and B be complex Banach algebras so that B is an algebraic Banach A-bimodule, i.e. it is a Banach A-bimodule and for given $a \in A$ and $b_1, b_2 \in B$ the following identities hold

$$a(b_1b_2) = (ab_1)b_2, (b_1b_2)a = b_1(b_2a), (b_1a)b_2 = b_1(ab_2).$$

Let $A \bowtie B$ denote the Cartesian product $A \times B$ endowed with the operations

$$z(a_1, b_1) + (a_2, b_2) = (za_1 + a_2, zb_1 + b_2),$$
$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + a_2 b_1 + b_1 b_2)$$

and the norm $||(a,b)||_1 = ||a|| + ||b||$ for any $z \in \mathbb{C}$, $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then $A \bowtie B$ becomes a complex Banach algebra, which is called *the generalized module extension Banach algebra* (or GMEBA) *of* A *and* B. These algebras were introduced to investigate questions concerning to weak amenability, generalizing and collecting similar structures such as classical module extension algebras, θ-Lau and T-Lau products, and direct sum of Banach algebras [2]. Advances in biprojectivity and biflatness of GMEBA's were obtained recently in [6]. Let us observe that $A \bowtie B$ has A and B-bimodule structures through the following respective operations

$$a'(a,b) = (a', 0_B)(a,b),\ (a,b)a' = (a,b)(a', 0_B),$$
$$b'(a,b) = (0_A, b')(a,b),\ (a,b)b' = (a,b)(0_A, b').$$

Throughout this article we shall consider the natural projections $A \bowtie B \xrightarrow{p_A} A$, $A \bowtie B \xrightarrow{p_B} B$, and the natural injections $A \xrightarrow{\iota_A} A \bowtie B$, $B \xrightarrow{\iota_B} A \bowtie B$. In case that B were an algebraic Banach A-bimodule with unit e_B we shall also consider the bounded operators

$$r_B : A \bowtie B \to B,\ r_B(a,b) = ae_B + b,$$
$$s_A : A \to A \bowtie B,\ s_A(a) = (a, -ae_B).$$

Note that p_A and ι_A are bounded A-bimodule maps and ι_B is a bounded B-bimodule map. Further, if B is an algebraic Banach A-bimodule with unit e_B so that $ae_B = e_B a$ if $a \in A$ then r_B and s_A are a B-bimodule map and an A-bimodule map respectively (cf. [6], Lemma 2.1).

We need to recall the notions of regularity of bounded bilinear operators and Banach algebras, amenability of Banach algebras, dual Banach algebras and Connes-amenability of Banach algebras.
(i) Given a bounded bilinear operator $\pi : X \times Y \to Z$ between complex normed spaces let $\pi' : Z^* \times X \to Y^*$ so that

$$\langle y, \pi'(z', x)\rangle = \langle \pi(x,y), z'\rangle \text{ if } x \in X, y \in Y, z' \in Z^*.$$

Besides let $\pi^t : Y \times X \to Z$ so that $\pi^t(y,x) = \pi(x,y)$ if $x \in X, y \in Y$. We shall say that π''' and $[(\pi^t)''']^t$ are *the first and second Arens extensions of* π. Then π is called *regular* (or *Arens regular*) if $\pi''' = [(\pi^t)''']^t (: X^{**} \times Y^{**} \to Z^{**})$ [1]. A Banach algebra A is called *regular* (or *Arens regular*) if the bounded bilinear operator $A \times A \xrightarrow{\pi} A, (a_1, a_2) \to a_1 a_2$ is regular. The second conjugate space

A^{**} of A, endowed with the products π''' or $[(\pi^t)''']^t$, become Banach algebras which contain a closed subalgebra isometrically isomorphic to A.
(ii) A Banach algebra A is called *amenable* if any bounded derivation $A \xrightarrow{D} X^*$ into the dual space of any Banach A-bimodule X is *inner*, i.e. there exists $\chi \in X^*$ so that $D(a) = a\chi - \chi a$ if $a \in A$ [7].
(iii) Following [12], we say that a Banach algebra A is a *dual Banach algebra* if there is a closed A-submodule A_* of A^* and an isomorphism of Banach A-bimodules $A \approx (A_*)^*$. Then it is said that A_* is a *predual* of A, which may be non unique.
(iv) Let A be a dual Banach algebra and let X be a Banach A-bimodule. Then we call X^* a w^*- *Banach* A-bimodule if for every $x' \in X^*$ the maps $a \mapsto ax'$ and $a \mapsto x'a$ are w^*-continuous from A into X^*.
(v) A dual Banach algebra A is *Connes-amenable* if every w^*-continuous derivation of A with values in a w^*-Banach A-bimodule X^* is inner [11].

2 Contents and main results

In Theorem 3.1 we shall determine conditions of amenability of GMEBA's. Preduality issues are considered in Section 4. Theorem 4.2 will establish conditions under which GMEBA's of dual Banach algebras are dual Banach algebras. In Section 5 we shall study Connes-amenability of the second dual space of GMEBA's. To this end Proposition 5.1 determines conditions of regularity of GMEBA's and Proposition 5.3 establish conditions of under which a GMEBA is an ideal in its second conjugate space. Theorem 5.1 will precise conditions of Connes-amenability of the second dual of a GMEBA. Finally, we close Section 6 posing three problems derived from this research.

3 On amenability of GMEBA's

Theorem 3.1 *Let A and B be complex Banach algebras, and let B be an algebraic Banach A-bimodule endowed of a unit e_B so that $ae_B = e_Ba$ for all $a \in A$. Then $A \bowtie B$ is amenable if and only if A and B are amenable.*

Proof 3.1 *($\Rightarrow$) Let us write $L = A \bowtie B$ and let*

$$\pi_L : L \times L \to L \text{ so that } \pi_L(u, v) = uv \text{ if } u, v \in L.$$

It is induced a unique $\hat{\pi}_L \in B(L\hat{\otimes}L, L)$, where $\hat{\otimes}$ denotes the usual projective tensor product.
By hypothesis L is amenable, which is equivalent to say that L has a virtual diagonal. So there exists $M_L \in (L\hat{\otimes}L)^{**}$ so that $\hat{\pi}_L^{**}(M_L)u = u$ in L^{**} and $uM_L = M_Lu$ in $(L\hat{\otimes}L)^{**}$ for all $u \in L$ (cf. [8], Theorem 1.3).

First, let us see that $M_A = (p_A \otimes p_A)^{**}(M_L)$ *is a virtual diagonal of* A*. For,* $M_A \in (A\hat{\otimes}A)^{**}$ *and* $\hat{\pi}_A^{**}(M_A) = (\hat{\pi}_A \circ (p_A \otimes p_A))^{**}(M_L)$*. We see that*

$$\begin{aligned}(\hat{\pi}_A \circ (p_A \otimes p_A))((a,b)\otimes(a',b')) &= \hat{\pi}_A(a\otimes a')\\ &= aa'\\ &= p_A((aa', ab'+ba'+bb'))\\ &= p_A((a,b)(a',b'))\\ &= (p_A\circ\hat{\pi}_L)((a,b)\otimes(a',b'))\end{aligned}$$

on simple tensors. So, $\hat{\pi}_A^{**}(M_A) = p_A^{**}(\hat{\pi}_L^{**}(M_L))$ *and as* $p_A \in_A Hom_A(L,A)$ *given* $a \in A$ *we see that*

$$\begin{aligned}\hat{\pi}_A^{**}(M_A)a &= p_A^{**}(\hat{\pi}_L^{**}(M_L))a\\ &= p_A^{**}(\hat{\pi}_L^{**}(M_L)a)\\ &= p_A^{**}(\kappa_L(a,0_B))\\ &= \kappa_A(a),\end{aligned}$$

where $\kappa_L : L \hookrightarrow L^{**}$ *and* $\kappa_A : A \hookrightarrow A^{**}$ *are the respective natural isometric embeddings of* L *and* A *into their second dual spaces. Besides, given* $a \in A$ *we have that*

$$\begin{aligned}aM_A &= a(p_A\otimes p_A)^{**}(M_L)\\ &= (p_A\otimes p_A)^{**}((a,0_B)M_L)\\ &= (p_A\otimes p_A)^{**}(M_L(a,0_B))\\ &= (p_A\otimes p_A)^{**}(M_L)a\\ &= M_A a\end{aligned}$$

and A *becomes amenable.*
Secondly, we shall prove that B *is amenable. Now, let* $M_B = (r_B\otimes r_B)^{**}(M_L)$ *in* $(B\hat{\otimes}B)^{**}$*. Then* $\hat{\pi}_B^{**}(M_B) = (\hat{\pi}_B\circ(r_B\otimes r_B))^{**}(M_L)$*. Further,*

$$\begin{aligned}(\hat{\pi}_B\circ(r_B\otimes r_B))((a_1,b_1)\otimes(a_2,b_2)) &= \hat{\pi}_B((a_1e_B+b_1)\otimes(a_2e_B+b_2))\\ &= a_1a_2e_B + a_1b_2 + b_1a_2 + b_1b_2\\ &= r_B(a_1a_2, a_1b_2+b_1a_2+b_1b_2)\\ &= (r_B\circ\hat{\pi}_L)((a_1,b_1)\otimes(a_2,b_2))\end{aligned}$$

on simple tensors. Hence $\hat{\pi}_B^{**}(M_B) = (r_B\circ\hat{\pi}_L)^{**}(M_L)$*. Since* $r_B \in_B Hom_B(L,B)$ *if* $b \in B$ *we have*

$$\begin{aligned}\hat{\pi}_B^{**}(M_B)b &= r_B^{**}(\hat{\pi}_L^{**}(M_L))b\\ &= r_B^{**}(\hat{\pi}_L^{**}(M_L)(0_A,b))\\ &= r_B^{**}(\kappa_L(0_A,b))\\ &= \kappa_B(r_B(0_A,b))\\ &= \kappa_B(b).\end{aligned}$$

Also, given $\beta \in (B\hat{\otimes}B)^*$ *the following equalities hold*

$$\begin{aligned}(r_B \otimes r_B)^*(\beta b) &= (r_B \otimes r_B)^*(\beta)b, \\ (r_B \otimes r_B)^*(b\beta) &= b(r_B \otimes r_B)^*(\beta).\end{aligned}$$

For instance, for the first one it suffices to note that

$$\begin{aligned}\langle (a_1, b_1) \otimes (a_2, b_2)), (r_B \otimes r_B)^*(\beta b)\rangle &= \langle (a_1 e_B + b_1) \otimes (a_2 e_B + b_2), \beta b\rangle \\ &= \langle (ba_1 + bb_1) \otimes (a_2 e_B + b_2), \beta\rangle \\ &= \langle (0_A, ba_1 + bb_1) \otimes (a_2, b_2), (r_B \otimes r_B)^*(\beta)\rangle \\ &= \langle (a_1, b_1) \otimes (a_2, b_2)), (r_B \otimes r_B)^*(\beta)b\rangle\end{aligned}$$

on simple tensors. Consequently if $b \in B$ *we see that*

$$\begin{aligned}\langle \beta, bM_B\rangle &= \langle \beta b, (r_B \otimes r_B)^{**}(M_L)\rangle \\ &= \langle (r_B \otimes r_B)^*(\beta b), M_L\rangle \\ &= \langle (r_B \otimes r_B)^*(\beta)b, M_L\rangle \\ &= \langle (r_B \otimes r_B)^*(\beta), (0_A, b))M_L\rangle \\ &= \langle (r_B \otimes r_B)^*(\beta), M_L(0_A, b)\rangle \\ &= \langle b(r_B \otimes r_B)^*(\beta), M_L\rangle \\ &= \langle (r_B \otimes r_B)^*(b\beta), M_L\rangle \\ &= \langle b\beta, M_B\rangle \\ &= \langle \beta, M_B b\rangle\end{aligned}$$

and B *becomes amenable.*
($\Leftarrow$) *The assertion will follow if we prove that* L *has a bounded approximate identity and* $\hat{\pi}_L^* \in_L Hom_L(L^*, (L\hat{\otimes}L)^*)$ *has a left inverse wich is a morphism of* L*-bimodules (cf. [9], Theorem 1).*
For, as A *is amenable it has a bounded approximate identity* $\{a_i\}_{i\in I}$ *(cf. [10], II, Theorem 21, II). Given* $i \in I$ *let* $l_i = (a_i, e_B - a_i e_B)$ *in* L*. Clearly* $\{l_i\}_{i\in I}$ *is bounded and if* $(a, b) \in L$ *we have*

$$\begin{aligned}l_i(a, b) &= (a_i, e_B - a_i e_B)(a, b) \\ &= (a_i a, a_i b + (ae_B - a_i a e_B) + (b - a_i b)) \\ &= (a_i a, (ae_B - a_i a e_B) + b) \\ &\xrightarrow{i\in I} (a, b).\end{aligned}$$

Analogously $(a, b)l_i \xrightarrow{i\in I} (a, b)$ *and* $\{l_i\}_{i\in I}$ *becomes a bounded approximate identity of* L*.*
Now, let $\theta_A \in_A Hom_A((A\hat{\otimes}A)^*, A^*)$ *and* $\theta_B \in_B Hom_B((B\hat{\otimes}B)^*, B^*)$ *be left inverses of* $\hat{\pi}_A^*$ *and* $\hat{\pi}_B^*$ *respectively. It is straightforward to see that*

$$\hat{\pi}_L \circ (s_A \otimes s_A) = s_A \circ \hat{\pi}_A \text{ and } \hat{\pi}_L \circ (\iota_B \otimes \iota_B) = \iota_B \circ \hat{\pi}_B.$$

Therefore $\hat{\pi}_A^* \circ s_A^* = (s_A \otimes s_A)^* \circ \hat{\pi}_L^*$ *and* $\hat{\pi}_B^* \circ \iota_B^* = (\iota_B \otimes \iota_B)^* \circ \hat{\pi}_L^*$.
Hence the following identities hold

$$s_A^* = \theta_A \circ (s_A \otimes s_A)^* \circ \hat{\pi}_L^*, \tag{1}$$
$$\iota_B^* = \theta_B \circ (\iota_B \otimes \iota_B)^* \circ \hat{\pi}_L^*. \tag{2}$$

Let us write

$$\theta_L : (L \hat{\otimes} L)^* \to L^*,$$
$$\theta_L = p_A^* \circ \theta_A \circ (s_A \otimes s_A)^* + r_B^* \circ \theta_B \circ (\iota_B \otimes \iota_B)^*.$$

By (1) and (2) we see that

$$\begin{aligned} \theta_L \circ \hat{\pi}_L^* &= p_A^* \circ s_A^* + r_B^* \circ \iota_B^* \\ &= (s_A \circ p_A + \iota_B \circ r_B)^* \\ &= Id_L^* \\ &= Id_{L^*}. \end{aligned}$$

Now we shall see that θ_L *is an* L*-bimodule homomorphism. For, let us write*

$$\zeta_A = p_A^* \circ \theta_A \circ (s_A \otimes s_A)^*,$$
$$\zeta_B = r_B^* \circ \theta_B \circ (\iota_B \otimes \iota_B)^*,$$

and let $\Lambda \in (L \hat{\otimes} L)^*$, $(a, b), (a', b') \in L$, $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$. *Then*

$$\begin{aligned} \langle a_1 \otimes a_2, ((s_A \otimes s_A))^*((a, b)\Lambda) \rangle &= \langle (a_1, -a_1 e_B) \otimes (a_2, -a_2 e_B), (a, b)\Lambda \rangle \\ &= \langle (a_1, -a_1 e_B) \otimes (a_2 a, -a_2 a e_B), \Lambda \rangle \\ &= \langle (s_A \otimes s_A)(a_1 \otimes (a_2 a)), \Lambda \rangle \\ &= \langle (s_A \otimes s_A)(a_1 \otimes a_2)a, \Lambda \rangle \\ &= \langle a_1 \otimes a_2, a(s_A \otimes s_A)^*(\Lambda) \rangle, \end{aligned}$$

and we can infer that $(s_A \otimes s_A)^*((a, b)\Lambda) = a(s_A \otimes s_A)^*(\Lambda)$. *Consequently*

$$\begin{aligned} \langle (a', b'), \zeta_A((a, b)\Lambda) \rangle &= \langle a', \theta_A \circ (s_A \otimes s_A)^*)((a, b)\Lambda) \rangle \\ &= \langle a', a\theta_A (s_A \otimes s_A)^*(\Lambda) \\ &= \langle a'a, \theta_A (s_A \otimes s_A)^*(\Lambda) \\ &= \langle p_A((a', b')(a, b)), \theta_A (s_A \otimes s_A)^*(\Lambda) \rangle \\ &= \langle (a', b'), (a, b)\zeta_A(\Lambda) \rangle, \end{aligned}$$

i.e. ζ_A *is a left L-module homomorphism. Besides*

$$\begin{aligned}\langle b_1 \otimes b_2, (\iota_B \otimes \iota_B)^*((a,b)\Lambda)\rangle &= \langle (0_A, b_1) \otimes (0_A, b_2), (a,b)\Lambda\rangle \\ &= \langle (0_A, b_1) \otimes (0_A, b_2a + b_2b), \Lambda\rangle \\ &= \langle \iota_B(b_1) \otimes \iota_B(b_2 r_B(a,b)), \Lambda\rangle \\ &= \langle b_1 \otimes (b_2 r_B(a,b)), (\iota_B \otimes \iota_B)^*(\Lambda)\rangle \\ &= \langle b_1 \otimes b_2, r_B(a,b)(\iota_B \otimes \iota_B)^*(\Lambda)\rangle\end{aligned}$$

and we deduce that

$$(\iota_B \otimes \iota_B)^*((a,b)\Lambda) = r_B(a,b)(\iota_B \otimes \iota_B)^*(\Lambda) \tag{3}$$

Similarly,

$$(\iota_B \otimes \iota_B)^*(\Lambda(a,b)) = (\iota_B \otimes \iota_B)^*(\Lambda) r_B(a,b). \tag{4}$$

Further, it is straightforward to see that

$$\hat{\pi}_B \circ (r_B \otimes r_B) = r_B \circ \hat{\pi}_L. \tag{5}$$

By (3) and (5) we obtain that

$$\begin{aligned}\langle (a',b'), \zeta_B((a,b)\Lambda)\rangle &= \langle a'e_B + b', \theta_B((\iota_B \otimes \iota_B)^*((a,b)\Lambda)\rangle \\ &= \langle r_B(a',b') r_B(a,b), \theta_B((\iota_B \otimes \iota_B)^*(\Lambda))\rangle \\ &= \langle (\hat{\pi}_B \circ (r_B \otimes r_B))((a',b') \otimes (a,b)), \theta_B((\iota_B \otimes \iota_B)^*(\Lambda))\rangle \\ &= \langle (a',b') \otimes (a,b), (r_B \circ \hat{\pi}_L)^*(\theta_B((\iota_B \otimes \iota_B)^*(\Lambda)))\rangle \\ &= \langle (a',b')(a,b), \zeta_B(\Lambda)\rangle \\ &= \langle (a',b'), (a,b)\zeta_B(\Lambda)\rangle,\end{aligned}$$

i.e. ζ_B *is a left L-module homomorphism. Thus* θ_L *becomes a left L-module homomorphism.*
Analogously, as $(s_A \otimes s_A)^*(\Lambda(a,b)) = (s_A \otimes s_A)^*(\Lambda)a$ *then* ζ_A *becomes a right L-module homomorphism. Also, by (4) and (5)* ζ_B *also becomes a right L-module homomorphism. Thus* θ_L *is a right L-module homomorphism.*

4 Preduality of GMEBA's

Theorem 4.1 *[13] Let A be a dual Banach algebra with a predual* A_*. *Then any* w^**-closed subalgebra B of A is a dual Banach algebra.*

Proof 4.1 *Clearly B is closed because it is* w^**-closed. Let* $\chi_A : A \to (A_*)^*$ *be an isomorphism of Banach A-bimodules and let*

$$^{\perp}\chi_A(B) = \{a_* \in A_* : \langle a_*, \chi_A(b)\rangle = 0 \ \textit{if}\ b \in B\}.$$

It is plain that A_* *is a Banach* B*-bimodule and that* ${}^{\perp}\chi_A(B)$ *becomes a Banach* B*-submodule of* A_**. We define*

$$\omega : A_* \to B^*, \langle b, \omega(a_*)\rangle = \langle a_*, \chi_A(b)\rangle \text{ if } a_* \in A_*, b \in B. \tag{6}$$

Then ω *is a homomorphism of Banach* B*-modules with* $\ker(\omega) = {}^{\perp}\chi_A(B)$*. If we write* $B_* = \dfrac{A_*}{{}^{\perp}\chi_A(B)}$ *it is induced a well defined homomorphism of Banach* B*-modules*

$$\Omega : B_* \to B^*,\ \Omega(a_* + {}^{\perp}\chi_A(B)) = \omega(a_*) \text{ for all } a_* \in A_*.$$

Indeed, as Ω *is injective* B_* *becomes imbedded in* B^**. Now we define*

$$\chi_B : B \to (B_*)^*,\ \langle a_* + {}^{\perp}\chi_A(B), \chi_B(b)\rangle = \langle a_*, \chi_A(b)\rangle \text{ if } a_* \in A_*, b \in B. \tag{7}$$

If $b \in B$ *and* $a_* \in {}^{\perp}\chi_A(B)$ *by (6) is* $\langle a_*, \chi_A(b)\rangle = 0$ *because* $a_* \in \ker(\omega)$*. Thus* $\chi_B(b) : B_* \to \mathbb{C}$ *is well defined. It is clearly linear and by (7) for* $b \in B$ *we see that*

$$||\chi_B(b)|| \leq ||\chi_A(b)|| \leq ||\chi_A||\ ||b||, \tag{8}$$

i.e. $\chi_B(b) \in (B_*)^*$*. Also* χ_B *is clearly linear and by (8)* $||\chi_B|| \leq ||\chi_A||$*. It is now straightforward to see that* χ_B *is a Banach* B*-module homomorphism.*
By (7) we have $\ker(\chi_B) \subseteq \ker(\chi_A)$ *and so* χ_B *is injective.*
Finally, let us see that χ_B *is surjective. For, if* $n \in (B_*)^*$ *there exists* $a \in A$ *so that*

$$n(a_* + {}^{\perp}\chi_A(B)) = \langle a_*, \chi_A(a)\rangle \tag{9}$$

for all $a_* \in A$*. In particular, we observe that* $\chi_A(a) \in [{}^{\perp}\chi_A(B)]^{\perp}$*. By the Hahn-Banach theorem* $\chi_A(B)^{-w^*} = [{}^{\perp}\chi_{\mathcal{A}}(B)]^{\perp}$ *and* $\chi_A(a) \in \chi_A(B)$ *because* B *is* w^**-closed. Then* $a \in B$ *because* χ_U *is injective. By (7) and (9) we see that* $n = \chi_B(a)$ *and the result follows as consequence of the open mapping theorem.*

Theorem 4.2 *Let* $L = A \bowtie B$ *be the GMEBA of dual Banach algebras* A *and* B *with preduals* A_*, B_**, so that:*
(i) B *has a unit* e_B *so that* $ae_B = e_B a$ *for all* $a \in A$*.*
(ii) Any functional $\varphi(b_*) : a \to \langle ae_B, b_*\rangle$ *on* A *belongs to* A_* *if* $b_* \in B_*$*.*
(iii) There are isomorphisms $\chi_A : A \to (A_*)^*$, $\chi_B : B \to (B_*)^*$ *of Banach* A *and* B*-bimodules such that*

$$\langle \varphi(b_*), \chi_A(a)\rangle = \langle b_*, \chi_B(ae_B)\rangle \text{ if } a \in A, b_* \in B_*.$$

Then L *has a predual isometrically isomorphic to* $A_* \oplus_\infty B_*$*.*

Proof 4.2 *It is readily seen that*

$$\Gamma : L^* \to A^* \oplus_\infty B^*,$$
$$\Gamma(l') = (\iota_A^*(l'), \iota_B^*(l')),$$

defines an isometric isomorphism. Let us write $L_* = \Gamma^{-1}(A_* \oplus_\infty B_*)$.
Then L_* *is a closed subspace of* L^*. *Further,* L_* *is an* L*-bimodule because for any* a, b, a_*, b_* *the following identities hold:*

$$(a,b)\Gamma^{-1}(a_*, b_*) = \Gamma^{-1}(aa_* + \varphi(bb_*), ab_* + bb_*), \tag{10}$$
$$\Gamma^{-1}(a_*, b_*)(a,b) = \Gamma^{-1}(a_*a + \varphi(b_*b), b_*a + b_*b). \tag{11}$$

Now, there are $\chi_A : A \to (A_*)^*$ *and* $\chi_B : B \to (B_*)^*$ *isomorphisms of Banach* A *and* B*-bimodules, respectively. Given* $l \in L$ *and* $l_* \in L_*$ *we shall write*

$$\langle l_*, \chi_L(l)\rangle = \langle a_*, \chi_A(a)\rangle + \langle b_*, \chi_B(b)\rangle, \tag{12}$$

with $l = (a,b)$ *and* $l_* = \Gamma^{-1}(a_*, b_*)$. *It is plain that* $\chi_L(l)$ *is a linear form on* L_*. *Indeed, it is bounded and* $\| \chi_L(l) \| \leq \| \chi_A(a) \| + \| \chi_B(b) \|$. *Moreover,* $\chi_L : L \to (L_*)^*$, *it is linear and* $\| \chi_L \| \leq \| \chi_A \| + \| \chi_B \|$. *Besides, using (iii) and the identities (10) and (11) it follows that* χ_L *is an* L*-bimodule homomorphism.*
Let $l \in \ker(\chi_L)$. *If* $l = (a,b)$ *then* $\langle (a_*, 0_{B^*}), \chi_L(l)\rangle = 0$ *for all* $a_* \in A_*$. *By (12) we infer that* $\chi_A(a) = 0$ *and so* $a = 0_A$. *Analogously we see that* $b = 0_B$, *i.e.* χ_L *is a monomorphism.*
Lastly, if $v \in (L_*)^*$ *then* $(\Gamma^{-1} |_{A_* \oplus_\infty B_*})^*(v) \in (A_*)^* \oplus_1 (B_*)^*$. *Hence there is a unique* $(a_v, b_v) \in L$ *so that* $(\Gamma^{-1} |_{A_* \oplus_\infty B_*})^*(v) = (\chi_A(a_v), \chi_B(b_v))$. *Given* $l_* \in L_*$, $l_* = (a_*, b_*)$, *we can write*

$$\begin{aligned}\langle l_*, v\rangle &= \langle \Gamma^{-1}(a_*, b_*), v\rangle \\ &= \langle (a_*, b_*), (\Gamma^{-1} |_{A_* \oplus_\infty B_*})^*(v)\rangle \\ &= \langle (a_*, b_*), (\chi_A(a_v), \chi_B(b_v))\rangle \\ &= \langle l_*, \chi_L(a_v, b_v)\rangle,\end{aligned}$$

i.e. $v = \chi_L(a_v, b_v)$, χ_L *is epimorphism and by the open mapping theorem it becomes an isomorphism of Banach* L*-bimodules.*

5 Connes-amenability of $(A \bowtie B)^{**}$

In the following Propositions 5.1 and 5.3 we shall analize conditions of regularity of GMEBA's and conditions under which these algebras are ideals in their second conjugate spaces. For, let $L = A \bowtie B$ be the GMEBA of complex

Banach algebras A and B. We shall denote by $\Box_A$, $\Box_B$, $\Box_{AB}$, $\Box_{BA}$, $\Box_L$ and $\Diamond_A$, $\Diamond_B$, $\Diamond_{AB}$, $\Diamond_{BA}$, $\Diamond_L$ to the respective first and second Arens extensions associated to the bounded bilinear operators

$$\pi_A : A \times A \to A,\ \pi_A(a_1, a_2) = a_1 a_2, \tag{13}$$
$$\pi_B : B \times B \to B,\ \pi_B(b_1, b_2) = b_1 b_2, \tag{14}$$
$$\pi_{AB} : A \times B \to B,\ \pi_{AB}(a, b) = ab, \tag{15}$$
$$\pi_{BA} : B \times A \to B,\ \pi_{BA}(b, a) = ba. \tag{16}$$
$$\pi_L : L \times L \to L, \pi_L(l_1, l_2) = l_1 l_2. \tag{17}$$

Proposition 5.1 *The Arens regularity of L is equivalent to the joint Arens regularity of the actions (13)-(16).*

Proof 5.2 *We have that $L^{**} \approx A^{**} \oplus_1 B^{**}$ and for all $(a_1'', b_1''), (a_2'', b_2'') \in L^{**}$ it is straightforward to see that the following relations hold*

$$(a_1'', b_1'')\Box_L(a_2'', b_2'') = (a_1''\Box_A a_2'', a_1''\Box_{AB} b_2'' + b_1''\Box_{BA} a_2'' + b_1''\Box_B b_2''),$$
$$(a_1'', b_1'')\Diamond_L(a_2'', b_2'') = (a_1''\Diamond_A a_2'', a_1''\Diamond_{AB} b_2'' + b_1''\Diamond_{BA} a_2'' + b_1''\Diamond_B b_2'').$$

So, if L is regular it is immediate that A and B become regular. Picking out $b_1'' = 0_{B^{**}}$ (resp. $b_2'' = 0_{B^*}$) it follows that π_{AB} (resp. π_{BA}) is regular. On the other side, the condition is clearly sufficient.

Now, as a consequence of Gantmacher's theorem we get the following (cf. [5], Theorem VI.4.8).

Corollary 5.3 *Let us assume that L is regular. Then $\kappa_L(L) \trianglelefteq L^{**}$ (i.e. $\kappa_L(L)$ is an ideal in L^{**}) if and only if $\kappa_A(A) \trianglelefteq A^{**}$, $\kappa_B(B) \trianglelefteq B^{**}$, and the bilinear operators π_{AB} and π_{BA} are separately weakly compact (i.e. on fixing any variable they induce weakly compact operators in the remaining variable).*

Remark 5.4 *The second conjugate algebra A^{**} of any regular-amenable Banach algebra A is Connes amenable (cf. [11], Corollary 4.3). The converse holds if besides $\kappa_A(A) \trianglelefteq A^{**}$. (cf. [11], Theorem 4.4).*

Theorem 5.1 *Let L be the GMEBA of Banach algebras A and B, where B has a unit e_B so that $ae_B = e_B a$ for all $a \in A$. If L is regular and $\kappa_L(L) \trianglelefteq L^{**}$ then L^{**} is Connes-amenable if and only if A^{**} and B^{**} are Connes-amenable.*

Proof 5.5 *By Proposition 5.1 A and B become regular and by Proposition 5.3 is $\kappa_A(A) \trianglelefteq A^{**}$ and $\kappa_B(B) \trianglelefteq B^{**}$.*
As L is regular and $\kappa_L(L) \trianglelefteq L^{**}$, if L^{**} is Connes-amenable then L becomes amenable. Besides, by Theorem 3.1 A and B become amenable and the necessity follows by Remark 5.4.
Analogously, if A^{**} and B^{**} are Connes-amenable then A and B become amenable. So, by Theorem 3.1 L is amenable and the sufficiency follows by Remark 5.4.

6 Open Problems

Problem 6.1 *Let L be the GMEBA of Banach algebras A and B. If L is amenable then A becomes amenable. Is B necessarily amenable?-*

In Theorem 3.1 we gave a partial answer, in the frame of [6], by assuming that B has a unit e_B that commutes with every element of A.
Let B be a commutative Banach and let $T : A \to B$ be an algebra homomorphism so that $||T|| \leq 1$. Then a new product on L is induced if we write

$$(a, b) \cdot_T (a', b') = (aa', T(a)b' + bT(a') + bb')$$

for every $(a, b), (a', b') \in L$. Then we obtain an analogous Banach algebra structure $A \bowtie_T B$ to $A \bowtie B$ that is amenable if and only if A and B are amenable (cf. [4], Theorem 4.1) Let A be a dual Banach algebra with a predual A_*. Then $\kappa_{A_*}(A_*)^\perp$ becomes an algebraic Banach $\kappa_A(A)$-bimodule of A^{**} with respect to booth Arens products. Further, $A^{**} = \kappa_A(A) \bowtie \kappa_{A_*}(A_*)^\perp$ (cf. [3], Theorem 5.1). If A^{**} were a dual Banach algebra with predual A^* its multiplication should be separately w^*-continuous, for instance in case that A be a regular Banach algebra (cf. [11], Prop. 1.2.(i)). As mentioned, incidentally this observation shows that advent of GMEBA's is quite natural. Nevertheless, the relation between GMEBA's and dual Banach algebras is more subtle and leads to the following questions:

Problem 6.2 *More generally, let $A \bowtie B$ be a GMEBA that is dual Banach algebra with predual L_*. Under what conditions A and B would be dual Banach algebras?-*

Problem 6.3 *Further, let us suppose that $A \bowtie B$ is a Connes-amenable Banach algebra. What could be infer about A and B?-*

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