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On some new difference double sequence spaces via Orlicz function

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Abstract

In this short article, we discussed some difference double sequence spaces defined by Orlicz function and study different properties of these spaces and also establish some inclusion results among them.

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1 Introduction

Definition 1.1 *([1]) The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that*

$$k_o = 0,\ h_r = k_r - k_{r-1} \to \infty \quad as\ r \to \infty$$

and

$$l_o = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \quad as\ s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) :\ k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \bar{q}_s.$$

Definition 1.2 *([2]) A double sequence $x = (x_{k,l})$ has a* Pringsheim limit L *(denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "$P-$*convergent"*.Let $A = (a_{m,n,k,l})_{k,l=0}^{\infty}$ be a four dimensional infinite matrix of real numbers for all $m, n = 0, 1, \ldots$. The sums*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{m,n,k,l}x_{k,l} = \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$$

are called A-transforms of the double sequence $x = (x_{k,l})$. We say that a sequence $x = (x_{k,l})$ is A-summable to the limit L if the A-transform of $x = (x_{k,l})$ exists for all $m, n = 0, 1, \ldots$ and convergent in the Pringsheim sense, i.e.,

$$\lim_{p,q\to\infty} \sum_{k,l=0}^{p,q} a_{m,n,k,l}x_{k,l} = y_{m,n}$$

and

$$\lim_{m,n} y_{m,n} = s.$$

The four dimensional matrix A is said to be $RH-$*regular* if it maps every bounded $P-$*convergent* sequence into a $P-$*convergent* sequence with the same $P-$*limit*.The assumption of boundedness was made because a double sequence which is P-convergent is not necessarily bounded. Using this definition Robison and Hamilton, independently, both presented the following Silverman-Toeplitz type characterization of $RH-$*regularity*.

Theorem 1.1 *([3, 4]) The four dimensional matrix A is $RH-$*regular *if and only if*

$RH_1 : P - \lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
$RH_2 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1;$
$RH_3 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 0$ *for each l;*
$RH_4 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 0$,*for each k;*
$RH_5 : \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is $P-$*convergent*;*
$RH_6 :$ *There exist finite positive integers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A.$

Definition 1.3 *Let M be an Orlicz function, $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers, $A = (a_{m,n,k,l})$ be a nonnegative $RH-$*regular *summability matrix method and $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. We* *now define the following new double sequence spaces:*

$$w_o^u(A,M,p)_\Delta = \Big\{ x=(x_{k,l}) \in s^u : P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{m,n\in I_{r,s}} \left[M\left(\sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} \frac{|\Delta x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho>0 \Big\},$$

$$w^u(A,M,p)_\Delta = \Big\{ x=(x_{k,l}) \in s^u : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{m,n\in I_{r,s}} \left[M\left(\left| \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} \frac{|\Delta x_{k,l}-L|}{\rho} \right| \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho>0 \text{ and } L \Big\},$$

and

$$w_\infty^u(A,M,p)_\Delta = \Big\{ x=(x_{k,l}) \in s^u : \sup_{m,n,k,l} \frac{1}{h_{r,s}} \sum_{m,n\in I_{r,s}} \left[M\left(\sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} \frac{|\Delta x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty \Big\}.$$

When $M(x)=x$, we have the following difference sequence spaces:

$$w_o^u(A,p)_\Delta = \Big\{ x=(x_{k,l}) \in s^u : P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{m,n\in I_{r,s}} \left(\sum_{k,l=0}^{\infty} a_{m,n,k,l} \frac{|\Delta x_{k,l}|}{\rho} \right)^{p_{k,l}} = 0, \text{ for some } \rho>0 \Big\},$$

$$w^u(A,p)_\Delta = \Big\{ x=(x_{k,l}) \in s^u : P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{m,n\in I_{r,s}} \left(\sum_{k,l=0}^{\infty} a_{m,n,k,l} \frac{|\Delta x_{k,l}-L|}{\rho} \right)^{p_{k,l}} = 0, \text{ for some } \rho>0 \text{ and } L \Big\},$$

and

$$w_{\infty}^{u}(A,p)_{\Delta}=\Big\{x=(x_{k,l})\in s^{u}:\sup_{m,n,k,l}\frac{1}{h_{r,s}}\sum_{m,n\in I_{r,s}}\left(\left|\sum_{k,l=0}^{\infty,\infty}a_{m,n,k,l}x_{k,l}\right|\right)^{p_{k,l}}<\infty\Big\}.$$

Some spaces are defined by specializing A, f and $p=(p_{k,l})$. For example, if $A=(C,1,1)$ the difference sequence spaces defined above become $w_{o}^{u}(M,p)_{\Delta}$, $w^{u}(M,p)_{\Delta}$ and $w_{\infty}^{u}(M,p)_{\Delta}$ which are as follows:

$$w_{o}^{u}(M,p)_{\Delta}=\Big\{x=(x_{k,l})\in s^{u}:P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}=0,\ \text{for some } \rho>0\Big\},$$

$$w^{u}(M,p)_{\Delta}=\Big\{x=(x_{k,l})\in s^{u}:P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{|\Delta x_{k,l}-L|}{\rho}\right)\right]^{p_{k,l}}=0,\ \text{for some } \rho>0 \text{ and } L\Big\},$$

and

$$w_{\infty}^{u}(M,p)_{\Delta}=\Big\{x=(x_{k,l})\in s^{u}:\sup_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}<\infty,\ \text{for some } \rho>0\Big\}.$$

Let $A=(C,1,1)$, $p_{k,l}=1$, for all $k,l\in\mathbb{N}$ and $M(x)=x$, we obtain the following difference sequence spaces:

$$w_{o,\Delta}^{u}=\left\{x=(x_{k,l})\in s^{u}:P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}|\Delta x_{k,l}|=0\right\},$$

$$w_{\Delta}^{u}=\left\{x=(x_{k,l})\in s^{u}:P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}|\Delta x_{k,l}-L|=0,\ \text{for some } L\right\},$$

and

$$w_{\infty,\Delta}^{u}=\left\{x=(x_{k,l})\in s^{u}:\sup_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}|\Delta x_{k,l}|<\infty\right\}.$$

2 Main results

In this section we shall establish some basic properties for the difference sequence spaces defined above.

Theorem 2.1 *Let* $p = (p_{k,l})$ *be bounded. The classes of sequences* $w_o^u(A, M, p)_\Delta$, $w^u(A, M, p)_\Delta$ *and* $w_\infty^u(A, M, p)_\Delta$ *are linear spaces.*

Theorem 2.2 *If* $0 < h = \inf p_{k,l} \le \sup p_{k,l} = H < \infty$, *then for any Orlicz function* M *and a nonnegative* $RH-$*regular summability matrix method* A, *then* $w^u(A, p)_\Delta \subset w^u(A, M, p)_\Delta$.

Theorem 2.3 $w_o^u(A, M, p)_\Delta$,$w^u(A, M, p)_\Delta$ *and* $w_\infty^u(A, M, p)_\Delta$ *are complete linear topological spaces with the paranorm*

$$g((x_{k,l})) = \sup_k |x_{k,1}| + \sup_l |x_{1,l}| + \inf\left\{\rho^{\frac{p_{k,l}}{T}} > 0 : \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{|\Delta x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \le 1\right\}$$

where $T = \max(1, H)$, $H = \sup_{k,l} p_{k,l}$.

Proposition 2.1 ***(a)*** $w^u(A, M, p)_\Delta \subset w_\infty^u(A, M, p)_\Delta$, ***(b)*** $w_o^u(A, M, p)_\Delta \subset w_\infty^u(A, M, p)_\Delta$.

Theorem 2.4 *The spaces* $w_o^u(A, M, p)_\Delta$ *and* $w^u(A, M, p)_\Delta$ *are nowhere dense subsets of* $w_\infty^u(A, M, p)_\Delta$.

Theorem 2.5 ***(a)*** *If* $0 < h = \inf p_{k,l} < p_{k,l} \le 1$, *then*

$$w^u(A, M, p)_\Delta \subset w^u(A, M)_\Delta.$$

(b) *If* $1 \le p_{k,l} \le \sup p_{k,l} < \infty$, *then*

$$w^u(A, M)_\Delta \subset w^u(A, M, p)_\Delta.$$

Theorem 2.6 *If* $\sup \frac{p_{k,l}}{p_{i,j}} < \infty$ *for all* $k \ge i$, $l \ge j$, *then* $w^u(A, M, p) \subset w^u(A, M, p)_\Delta$ *and the inclusion is strict, where*

$$w^u(A, M, p) = \left\{x = (x_{k,l}) \in s^u : P - \lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{|x_{k,l} - L|}{\rho}\right)\right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \text{ and } L\right\}.$$

Example 2.2 *The inclusion is strict follows from the following example.*

Let $A = (C, 1, 1)$, $M(x) = x^p$, $p_{k,l} = 1$ *for all* k *odd and for all* $l \in \mathbb{N}$ *and* $p_{k,l} = 2$ *otherwise. Consider the sequence* $x = (x_{k,l})$ *defined by* $x_{k,l} = k + l$ *for all* $k, l \in \mathbb{N}$. *We have* $\Delta x_{k,l} = 0$ *for all* $k, l \in \mathbb{N}$. *Hence* $x = (x_{k,l}) \in w^u(A, M, p)_\Delta$ *but* $x = (x_{k,l}) \notin w^u(A, M, p)$.

3 Double Δ-Statistical Convergence

The concept of statistical convergence for single sequences was introduced by Fast [5] in 1951. Later, Mursaleen and Edely [6] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be *$P-$statistical convergence to L* provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \{\text{the number of } (k,l): \; k < m, l < n; \; |x_{k,l} - L| \geq \varepsilon\} = 0.$$

In this case, we write $st_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all *$P-$statistical convergent double sequences* by st_2.

Definition 3.1 *A real double sequence $x = (x_{k,l})$ is said to be $P-$*statistical $\Delta-$convergence to L *provided that for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} \{\textit{the number of } (k,l): \; k < m, l < n; \; |\Delta x_{k,l} - L| \geq \varepsilon\} = 0.$$

In this case, we write $st_{2,\Delta} - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-$statistical $\Delta-$convergent double sequences *by $st_{2,\Delta}$.*

Theorem 3.2 *If M be an Orlicz function, then $w^{u}(M)_{\Delta} \subset st_{2,\Delta}$.*

Theorem 3.3 *$st_{2,\Delta} = w_{o}^{u}(M)_{\Delta}$ if and only if the Orlicz function M is bounded.*

4 Open Problem

In this paper, we discussed some difference double sequence spaces defined by Orlicz function and study different properties of these spaces and also establish some inclusion results among them. It is open problem that what are the results of this study in more generalized sequence spaces?

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