Int. J. Open Problems Compt Math, Vol. 15, No. 4, December 2022
Print ISSN 1998-6262, Online ISSN: 2079-0376

A Brief Review about Fractional Calculus

Iqbal M. Batiha[1,2,*], Shameseddin Alshorm[1,*], IqbalL H. Jebril[1], and Mamon Abu Hammad[1]

[1] Department of Mathematics, Al Zaytoonah University of Jordan,
Queen Alia Airport St 594, Amman 11733, Jordan
e-mail: i.batiha@zuj.edu.jo (I.M.B), alshormanshams@gmail.com (S.A)
i.jebril@zuj.edu.jo (I.H.J)
[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE

Abstract

In this paper, we recall some basic facts and preliminaries related to fractional calculus by making a literature review for its definitions and properties. This would provide sufficient knowledge about this important topic. Several lemmas and theorems are shown in detail for completeness.

Keywords: *Fractional Calculus, Gamma function, Beta function, Mittag-Leffler function, Riemann-Liouville.*

1 Introduction

Fractional calculus indicates the integration or the differentiation of non-integer order. Interestingly enough, this topic has a long history in calculus. The first discussion of fractional calculus was between Leibniz and L'Hopital. He actually asked the latter about the differentiation of order half of certain functions. However, there are some mathematicians, like Riemann, Abel, Liouville, and Lacroix, who laid the foundations for fractional calculus and dominated the field. In his famous paper on the even time problem, Abel was the first one who gave a physical description of the integral system of order 1/2 see [1, 2, 3, 4, 5]. Indeed, this article went further for solving an integral equation. A fractional derivative was originally mentioned by Lacroix in a paper that was published in 1819. He applied fractional calculus to resolve an integral equation arising from the formulation of the tautochrone problem. Applications of the theory of partial calculus expanded greatly during the nineteenth and twentieth centuries see [19, 20, 21, 22, 23], and many contributors provided definitions of fractional derivatives and integrals see [9].

2 Basic facts

For completeness, we shall define the gamma function and beta function in this section and show some of their aspects. By concentrating on the theory of Mittag-Leffler functions, we can describe a variety of occurrences in a number of processes that expand or decay too slowly to be described by conventional functions like the exponential function and its backdrops.

2.1 Gamma function

Leonhard Euler, a Swiss mathematician, was the one who initially proposed the Gamma function in order to extend the factorization to erroneous values. Later, other eminent mathematicians such as Christoph Gudermann, Carl Friedrich Gauss, Adrien-Marie Legendre, Charles Hermit, Karl Werkstrasse, Joseph Liouville, and many others researched it because of its significance. One of the fundamental operations in fractional calculus is the gamma function. It belongs to the class of special transcendental functions. This function can appear in various areas and fields like asymptotic series, definite integral, hypergeometric series, Riemann-zeta function, number theory, and others see [6, 7, 8]. In what follows, we will introduce the definition of the Gamma function, and demonstrate some of its properties for completeness, for more applications and contributions for these definitions see [10, 11, 12, 13, 14].

Definition 2.1 *The Gamma function* Γ *is a function* Γ*:*$\mathbb{R}^+ \to \mathbb{R}^+$ *satisfying the following equation:*

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt. \tag{1}$$

The domain of the Gamma function can be expanded beyond the set $\mathbb{R}^+$ (for example to complex numbers). The rather limited formulation given above, on the other hand, is sufficient to answer the vast majority of several statistics problems involving the Gamma function.

In what follows, we aim to introduce some basic properties associated with the Gamma function. These properties would pave the way to understanding several concepts connected with fractional calculus.

1. For $\alpha \in \mathbb{R}$, we have:

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1).$$

Proof. By using Defintion 1, we have:

$$\Gamma(\alpha) = \int_0^\infty v^{\alpha-1} e^{-v} dv.$$

With the help of applying integration by parts, we get:

$$\Gamma(\alpha) = (\alpha - 1) \int_0^\infty v^{(\alpha-1)-1} e^{-v} dv,$$

which directly gives the desired result.

2. For $n \in \mathbb{N}$, we have: $\Gamma(n) = (n-1)!$.

   Proof. First of all, with the help using property 2, we can have:

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2).$$

   If we continue in this manner, we get:

$$\Gamma(n) = (n-1)(n-2)...(3)(2)(1),$$

   which yields the aimed result.

3. For $\alpha \notin \mathbb{Z}$, we have:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}.$$

   Proof. By using Definition 1, we have:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \int_0^\infty e^{-u} u^{1-\alpha-1} du,$$

   or

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty e^{-(t+u)} \left(\frac{t}{u}\right)^\alpha \frac{1}{t}\, dt\, du. \tag{2}$$

   Now, we use the substitutions $\tau = t + u$ and $w = \frac{t}{u}$. From the later assumption, we can have $t = wu$, which makes the first assumption to be as $u = \frac{\tau}{1+w}$. This immediately implies that $t = \frac{w\tau}{1+w}$. Now, by using the change of variables to the Equation 2, we get:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty e^{-\tau} w^\alpha \left(\frac{1+w}{w\tau}\right) |J|\, d\tau\, dw, \tag{3}$$

   where $|J|$ is the Jacobian matrix in which it can be determined by:

$$|J| = \begin{vmatrix} \frac{\partial v}{\partial \zeta} & \frac{\partial v}{\partial x} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial x} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+x} & \frac{-\zeta}{(1+x)^2} \\ \frac{x}{1+x} & \frac{\zeta}{(1+x)^2} \end{vmatrix} = \frac{\zeta}{(1+x)^2}.$$

From this point of view, the Equation 3 becomes:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty e^{-\tau} w^\alpha \left(\frac{1+w}{w\tau}\right) \frac{\tau}{(1+w)^2}\ d\tau\ dw,$$

or

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty \frac{w^{\alpha-1}}{1+w}\ e^{-\tau}\ d\tau\ dw.$$

Consequently, we have:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \frac{w^{\alpha-1}}{1+w} \left(\int_0^\infty e^{-\tau} d\tau\right)\ dw,$$

which implies:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \frac{w^{\alpha-1}}{1+w}\ dw,$$

or

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^1 \frac{w^{\alpha-1}}{1+w} dw + \int_1^\infty \frac{w^{\alpha-1}}{1+w} dw.$$

Now, let $w = \frac{1}{v}$. Then, we get:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^1 \frac{w^{\alpha-1}}{1+w} dw + \int_0^1 \frac{v^{-\alpha}}{1+v} dv. \tag{4}$$

Note that the above two integrals have the same region, which makes us to let $y = w = v$ in Equation 4 and obtains:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^1 \frac{y^{\alpha-1} + y^{-\alpha}}{1+y} dy.$$

By utilizing geometric series, we can have:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^1 \left(y^{\alpha-1} + y^{-\alpha}\right) \left[\sum_{k=0}^\infty (-1)^k y^k\right] dy,$$

or

$$\Gamma(\alpha)\Gamma(1-\alpha) = \sum_{k=0}^\infty (-1)^k \int_0^1 y^{k+\alpha-1} + y^{k-\alpha} dy.$$

This implies:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \sum_{k=0}^\infty (-1)^k \left[\frac{1}{k+\alpha} + \frac{1}{k-\alpha+1}\right],$$

which gives the expansion of the power series of $\frac{\pi}{\sin(\alpha\pi)}$.

4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

   Proof. We have:

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right).$$

   This leads by using third property to the following equality:

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi,$$

   which yields the aimed result.

5. For $n \in \mathbb{N}$, we have:

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\prod_{j=0}^{n-1}(j+\frac{1}{2}).$$

   Proof. The result is obtained by iterating a recursion formula. In other words, in view of first property, we can obtain:

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-1+\frac{1}{2}\right)\Gamma\left(n-1+\frac{1}{2}\right),$$

   i.e.,

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-1+\frac{1}{2}\right)\left(n-2+\frac{1}{2}\right)\Gamma\left(n-2+\frac{1}{2}\right).$$

   If we continue in this manner, we reach the following assertion:

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-1+\frac{1}{2}\right)\left(n-2+\frac{1}{2}\right)...\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right).$$

   This immediately gives the desired result.

2.2 Beta function

Calculus depends heavily on the beta function because of its tight relationship to the gamma function. Calculus allows for the reduction of numerous difficult integrals into formulas that incorporate the Beta function. According to the following formula, this function actually has a relationship to the Gamma function:

Definition 2.2 *The Beta function can be defined as follows:*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \tag{5}$$

where $p, q > 0$.

Due to its strong resemblance to the Gamma function, a generalization of the factorial function, the Beta function is crucial in calculus. It can simplify a large number of complex integral functions into simple integrals. However, it can be explained as follows using the Gamma function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{6}$$

where the Gamma function is previously defined in Equation 2.

In the following content, we will demonstrate some basic properties connected with the Beta function with their proofs.

1. For $p, q \in \mathbb{R}^+$, we have: $B(p,q) = B(q,p)$.

   Proof. By using the Equation 6, we can have:

   $$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} = B(q,p).$$

2. For $p, q \in \mathbb{R}^+$, we have:

   $$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$$

   Proof. In order to prove this result, one can use first property, i.e.,

   $$B(p,q) = \int_0^1 x^{q-1}(1-x)^{p-1} dx. \tag{7}$$

   By using the substitution $x = \frac{1}{1+y}$, we can obtain:

   $$B(p,q) = \int_\infty^0 \frac{1}{(1+y)^{q-1}} \left(1 - \frac{1}{1+y}\right)^{p-1} \left(\frac{-1}{(1+y)^2}\right) dy.$$

   This consequently implies:

   $$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}} dy.$$

   Now, by replacing y instead of x, we obtain the wanted result.

3. $B(\frac{1}{2}, \frac{1}{2}) = \pi$.

   Proof. With the help of using Equation 6, we can have:

   $$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2})},$$

   which immediately gives:

   $$B\left(\frac{1}{2}, \frac{1}{2}\right) = (\sqrt{\pi})^2 = \pi.$$

2.3 Mittag-Leffler function

In the solutions of fractional differential equations, the Mittag-Leffler function plays an important role and appears frequently. The scientific community has recently become interested in the Mittag-Leffler functions as a result of the increased interest of researchers and scholars in both pure and applied mathematics as well as non-traditional models. We may describe a variety of occurrences in a range of processes that expand or decay too slowly to be described by standard functions like the exponential function and its backdrops by focusing on the theory of Mittag-Leffler functions.

Definition 2.3 *Let $\alpha > 0$. The Mittag-Leffler function $E_\alpha(\cdot)$ is defined by:*

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}. \tag{8}$$

It should be noted that the above series should be convergent. This function was introduced by Mittag-Leffler. It can be immediately noticed from the previous definition the following notation:

$$E_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x, \tag{9}$$

which represents the well-known exponential function. The following definition can be used to define the Mittag-Leffler function in its more general version.

Definition 2.4 *Let $\alpha, \beta > 0$. The function $E_{\alpha,\beta}(\cdot)$ defined by:*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \tag{10}$$

whenever the series converges is called the two-parameter Mittag-Leffler function with parameters α and β.

In light of Equation 10, and by taking the parameters α and β at specific values, one can generate several consecutive Mittag-Leffler functions. These functions are listed below for completeness:

$$E_{0,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(0+1)} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$$

$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x},$$

$$E_{1,3}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{1}{x^2} \sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)!} = \frac{e^x - 1 - x}{x^2}.$$

In general, we can have:

$$E_{1,m}(x) = \frac{1}{x^{m-1}} \left(e^x - \sum_{k=0}^{m-2} \frac{x^k}{k!} \right). \tag{11}$$

In the same regard, the two hyperbolic functions, the $\frac{\sinh(x)}{x}$ and $\cosh(x)$ functions, are also deemed as some particular cases of the Mittag-Leffler function. In other words, we can have the following two results:

$$E_{2,1}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh(x),$$

and

$$E_{2,2}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+2)} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \frac{\sinh(x)}{x}.$$

Next, we intend to introduce some primary properties associated with Mittag-Leffler function. These properties would play a major role in understanding the Mittag-Leffler function and how it can be employed in several implementations.

1. For $\alpha, \beta \in \mathbb{R}^+$, we have:

$$E_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + x E_{\alpha,\alpha+\beta}(x).$$

Proof. By Defintion 2.4, we can have:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} = \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}.$$

This is equivalent to say that:

$$E_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{\Gamma(\alpha(k+1) + \beta)},$$

which consequently leads us to the desired result.

2. For $\alpha, \beta \in \mathbb{R}^+$, we have:

$$\frac{d}{dx} E_{\alpha,\beta}(x) = \frac{E_{\alpha,\beta-1}(x) - (\beta - 1) E_{\alpha,\beta}(x)}{\alpha x}.$$

Proof. By taking the term $\beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(x)$, we can obtain:

$$\beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(x) = \beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta + 1)}.$$

This yields:

$$\beta E_{\alpha,\beta+1}(x)+\alpha x\frac{d}{dx}E_{\alpha,\beta+1}(x)=\beta\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta+1)}+\sum_{k=0}^{\infty}\frac{\alpha k x^k}{\Gamma(\alpha k+\beta+1)},$$

or

$$\beta E_{\alpha,\beta+1}(x)+\alpha x\frac{d}{dx}E_{\alpha,\beta+1}(x)=\sum_{k=0}^{\infty}\frac{(\alpha k+\beta)x^k}{(\alpha k+\beta)\Gamma(\alpha k+\beta)}=E_{\alpha,\beta}(x).$$

In other words, we have:

$$E_{\alpha,\beta}(x)=\beta E_{\alpha,\beta+1}(x)+\alpha x\frac{d}{dx}E_{\alpha,\beta+1}(x),$$

i.e.,

$$\frac{d}{dx}E_{\alpha,\beta+1}(x)=\frac{E_{\alpha,\beta}(x)-\beta E_{\alpha,\beta+1}(x)}{\alpha x},$$

which immediately gives the result.

3 Differentiation and integration operators

In this section, a literature review, main definitions, and theorems will be introduced for the Riemann-Liouville fractional-order integrator, Riemann-Liouville fractional-order differentiator, and Caputo fractional-order derivative differentiator.

3.1 Riemann-Liouville fractional-order integrator

Fractional calculus and its popular applications have been increasingly paid in very recent times. There are many known forms of fractional integral operators, and some of them have been extensively studied with their applications, the Riemann-Liouville operator is one of them. Also, it relates to a real function $f:\mathbb{R}\to\mathbb{R}$ with another function $J_a^\alpha f$ for each value of the parameter $\alpha > 0$, where a is the starting point of the definite integral. From the fact that asserts the integral is just a manner of generalization for several repeated antiderivatives of the function f, the Riemann–Liouville fractional-order integral operator $J_a^\alpha f$ is also an iterated antiderivative of the function f of order α. Actually, the Riemann-Liouville integral operator is named after two scientists Bernard Riemann and Joseph Liouville, for more see [15, 16, 17]. Next, we illustrate the definition of this operator coupled with some applications.

Definition 3.1 *Let α be a real nonnegative number. Then J_a^α defined on $L_1[a,b]$ by:*

$$J_a^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ a\le x\le b, \tag{12}$$

is called the Riemann-Liouville fractional-order integral operator of order α.

By referring to Equation 12, we can conclude the following property:

$$J_0^{\alpha} K = \frac{K}{\Gamma(\alpha+1)} x^{\alpha}, \tag{13}$$

where K is constant. In addition, if one assumes $\alpha = 0$ in Equation 12, then $J_a^0 = I$ will be yielded, which is called Identity operator. In the same regard, if we take $f(x) = (x-a)^p$ in Equation 12, where $a, p \in \mathbb{R}$, then we obtain:

$$J_a^{\alpha}(x-a)^p = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \cdot (t-a)^p dt.$$

By using the substitution $t = a + y(x-a)$, we can get:

$$J_a^{\alpha}(x-a)^p = \frac{1}{\Gamma(\alpha)} \int_0^1 (x-(a+y(x-a)))^{\alpha-1} \cdot (a+y(x-a)-a)^p (x-a) dy.$$

This yields:

$$J_a^{\alpha}(x-a)^p = \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} \int_0^1 y^p (1-y)^{\alpha-1} dy.$$

In other words, we have:

$$J_a^{\alpha}(x-a)^p = \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} B(p+1, \alpha),$$

or

$$J_a^{\alpha}(x-a)^p = \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} \cdot \frac{\Gamma(p+1)\Gamma(\alpha)}{\Gamma(\alpha+p+1)}.$$

This consequent yields:

$$J_a^{\alpha}(x-a)^p = \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} (x-a)^{\alpha+p}, \tag{14}$$

which is called the Power Rule property.

In what follows, we present some properties of the Riemann-Liouville fractional-order integral operator.

1. Let $m, n \geq 0$ and $f \in L_1[a, b]$. Then we have: $J_a^m J_a^n f = J_a^{m+n} f$, where $L_1[a, b]$ the set of all functions such that their absolute values are integrable on $[a, b]$.

   Proof. By using Equation 12, we can get:

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-t)^{m-1} \int_{\tau}^x (t-\tau)^{n-1} f(\tau) \cdot d\tau \; dt,$$

or

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau) \int_\tau^x (x-t)^{m-1} \cdot (t-\tau)^{n-1} \cdot dt\ d\tau.$$

Now, let $t = \tau + s(x-\tau)$, then we obtain:

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau) \int_0^1 (x - (\tau + s(x-\tau)))^{m-1} \cdot (\tau + s(x-\tau) - \tau)^{n-1} \cdot (x-\tau) \cdot ds\ d\tau,$$

or

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-\tau)^{m+n-1} \cdot f(\tau) \int_0^1 (1-s)^{m-1} \cdot s^{n-1} \cdot ds\ d\tau.$$

In other words, we have:

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x (x-\tau)^{m+n-1} \cdot f(\tau) \cdot B(n,m) \cdot d\tau.$$

This immediately implies:

$$J_a^m J_a^n f(x) = \frac{1}{\Gamma(m+n)} \int_a^x (x-\tau)^{m+n-1} \cdot f(\tau) \cdot d\tau = J_a^{m+n} f,$$

which yields the aimed result.

2. For $m, n \geq 0$, we have: $J_a^m J_a^n f = J_a^n J_a^m f$.

   Proof. By using the first property, we can obtain:

$$J_a^m J_a^n f = J_a^{m+n} f = J_a^{n+m} f = J_a^n J_a^m f.$$

3. For $m, n \geq 0$, we have: $J_a^{m+n} f = J_a^{m+n-1} J_a^1 f$.

   Proof. In accordance with the second property, we can gain:

$$J_a^{m+n} f = J_a^{m+n+1-1} f = J_a^{m+n-1} J_a^1 f.$$

3.2 Riemann-Liouville fractional-order differentiator

For many applications, the RLF differential operator technique is crucial. In actuality, RLF-order derivative operators are special instances of practically all other formulations of the fractional-order derivative operators. It is known that the fractional-order derivative of the constant is not zero, in contrast to standard calculus. The essential definition of the fractional-order Riemann-Liouville derivative operator is then given, followed by a few fundamental features.

Definition 3.2 *Let $\alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The operator D_a^α defined by:*

$$D_a^\alpha f = D^m J_a^{m-\alpha} f, \tag{15}$$

is called the Riemann-Liouville fractional-order differential operator of order α.

It should be remarked here that when $\alpha = 0$ in Equation 15, we gain $D_a^0 = I$, which is called Identity operator. Next, we aim to introduce the main definition of the operator at hand.

Definition 3.3 *Let α be a real nonnegative number. For a positive integer m such that $m-1 < \alpha \leq m$, the Riemann-Liouville fractional-order differential operator of a function f of order α is defined by:*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-t)^{m-\alpha-1} f(t) dt. \tag{16}$$

Without loss of generality, one might consider $a = 0$ in Equation 16 to obtain:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t) dt. \tag{17}$$

In addition, when $0 < \alpha \leq 1$, then the Riemann-Liouville fractional-order derivative of the function f of order α is defined by:

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t) dt. \tag{18}$$

From Equation 18, one can conclude the Power Rule property which can be outlined as follows:

$$D^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, \tag{19}$$

where $a, p \in \mathbb{R}$. In this regard, it should be observed that the fractional-order derivative of a constant function K is not zero. This means:

$$D^\alpha K = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} K, \tag{20}$$

where K is constant.

The following section provides various Riemann-Liouville fractional-order derivative operator properties along with their justifications. These properties can demonstrate the compositions of such operator with itself and with the Riemann-Liouville integral operator.

1. Let $\beta \geq 0$. Then for every $f \in L_1[a,b]$, we have $D_a^\beta J_a^\beta f = f$ almost every where. **Proof**. Let $m = \lceil \beta \rceil$, then by using Equation 15, we can get:

$$D_a^\beta J_a^\beta f = D^m J_a^{m-\beta} J_a^\beta f = D^m J_a^m f = D^{\lceil \beta \rceil} J_a^{\lceil \beta \rceil} f = f.$$

2. Let α_1,$\alpha_2 \geq 0$ and $\phi \in L_1[a,b]$ such that $f = J_a^{\alpha_1+\alpha_2}\phi$. Then $D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1+\alpha_2} f$, where $L_1[a,b]$ the set of all functions such that their absolute values are integrable on $[a,b]$.

Proof. Since $f = J_a^{\alpha_1+\alpha_2}\phi$, then we have:

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1} D_a^{\alpha_2} J_a^{\alpha_1+\alpha_2}\phi$$

But $D_a^{\alpha_1} = D^{\lceil \alpha_1 \rceil} J_a^{\lceil \alpha_1 \rceil - \alpha_1}$ and $D_a^{\alpha_2} = D^{\lceil \alpha_2 \rceil} J_a^{\lceil \alpha_2 \rceil - \alpha_2}$. Therefore, we can obtain:

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1} D_a^{\alpha_2} J_a^{\alpha_1+\alpha_2}\phi = D^{\lceil \alpha_1 \rceil} J_a^{\lceil \alpha_1 \rceil - \alpha_1} D^{\lceil \alpha_2 \rceil} J_a^{\lceil \alpha_2 \rceil - \alpha_2} J_a^{\alpha_1+\alpha_2}\phi,$$

or

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D^{\lceil \alpha_1 \rceil} J_a^{\lceil \alpha_1 \rceil - \alpha_1} J_a^{\alpha_1}\phi,$$

consequently, we have:

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D^{\lceil \alpha_1 \rceil} J_a^{\lceil \alpha_1 \rceil}\phi = \phi.$$

This implies:

$$D_a^{\alpha_1} D_a^{\alpha_2} f = \phi. \tag{21}$$

Now, since $f = J_a^{\alpha_1+\alpha_2}\phi$, then

$$D_a^{\alpha_1+\alpha_2} f = D_a^{\alpha_1+\alpha_2} J_a^{\alpha_1+\alpha_2}\phi = D^{\lceil \alpha_1+\alpha_2 \rceil} J_a^{\lceil \alpha_1+\alpha_2 \rceil - (\alpha_1+\alpha_2)} J_a^{\alpha_1+\alpha_2}\phi,$$

i.e.,

$$D_a^{\alpha_1+\alpha_2} f = D^{\lceil \alpha_1+\alpha_2 \rceil} J^{\lceil \alpha_1+\alpha_2 \rceil}\phi = \phi.$$

This implies:

$$D_a^{\alpha_1+\alpha_2} f = \phi. \tag{22}$$

By Equation 21 and Equation 22, we get:

$$D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1+\alpha_2} f.$$

This immediately gives the desired result.

3.3 Caputo fractional-order derivative differentiator

The RLF-order derivative operator is recognized to have a number of issues. First off, a constant's fractional-order derivative operator is not zero. Second, and more crucially, one must determine the following expression to find the Laplace transform of such an operator:

$$\lim_{x\to 0} D^{\alpha-1} f(x),$$

which unfortunately has no physical meaning. Form this point of view, different definitions of the fractional-order derivative operators have been then proposed, including the Caputo fractional-order derivative operator. This operator is given in the next two definitions by using the same notations used by podlubny in [24, 25, 26].

Definition 3.4 *Let $\alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The Caputo fractional-order derivative operator D_a^α is defined by:*

$$D_a^\alpha f = J_a^{m-\alpha} D^m f. \tag{23}$$

Definition 3.5 *Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$ such that $m - 1 < \alpha \leq m$. Then the Caputo fractional-order deivative operator of order α is given by:*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \ x > a. \tag{24}$$

It should be noted that if $a = 0$ in Equation 24, one can get the most reliable version of the Caputo fractional-order derivative operator. That is;

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \ x > 0. \tag{25}$$

In fact, there are a variety of outcomes that can be obtained from using the Caputo fractional-order operator. These outcomes are:

- The factorial-order derivative operator of a constant is always zero for all orders $\alpha > 0$.
- As $\alpha \to m$, $D^\alpha f(x) = D^m f(x) = f^{(m)}(x)$.
- The Laplace transform can be taken over $D_*^\alpha f(x)$. This is actually the main explanation behind the use of the Caputo fractional-order derivative operator in several physical implementations.

In view of the Equation 25, the Caputo operator formula that represents the Power Rule property can be obtained in the following manner:

$$D_*^{\alpha} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} & , \ m-1 < \alpha \leq m, \ p > m-1, \ p \in \mathbb{R} \\ 0 & , \ m-1 < \alpha \leq m, \ p \leq m-1, \ p \in \mathbb{N}. \end{cases} \tag{26}$$

As a result, we can also deduce the following constant formula as:

$$D_*^{\alpha} k = 0, \tag{27}$$

where K is constant.

Next, we present some basic properties related to the Caputo fractional-order derivative operator. These properties are related to the linearity and the non-commutation of the operator under consideration.

1. Let $m-1 < \alpha \leq m$ such that $m \in \mathbb{N}$. Then we have: $D_a^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D_a^{\alpha} f(x) + \mu D_a^{\alpha} g(x)$, where λ, μ are two scalers.

   Proof. With the help of using the Equation 24, we can get:

   $$D_a^{\alpha}(\lambda f(x) + \mu g(x)) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} (\lambda f^{(m)}(t) + \mu g^{(m)}(t)) dt.$$

   This implies:

   $$D_a^{\alpha}(\lambda f(x) + \mu g(x)) = \frac{\lambda}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt + \frac{\mu}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} g^{(m)}(t) dt,$$

   which means:

   $$D_a^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D_a^{\alpha} f(t) + \mu D_a^{\alpha} g(t).$$

2. Let $m-1 < \alpha \leq m$ such that $m, \beta \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then we have: $D_a^{\alpha} D_a^{\beta} f(x) = D_a^{\alpha+\beta} f(x) \neq D_a^{\beta} D_a^{\alpha} f(x)$.

4 Fractional differential equations (FDE)

In the last years, FDEs and their applications have received wide attention from many researchers. It is known that the ordinary differential equation is a special case of the FDEs. In particular, fractional derivative formulations have gained great importance and interest in many fields because of their applications in science and engineering. They are broadly used in mathematics, chemistry, physics, mechanics, medicine, biology, control theory, signal and image processing, environmental and financial sciences, and other various disciplines. In this connection, we present next an official definition of the fractional differential equation in view of the RL and Caputo fractional-order derivative operators, respectively.

Definition 4.1 *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $n \in \mathbb{N}$ such that $n-1 < \alpha \leq n$ and $g : A \subseteq \mathbb{R} \to \mathbb{R}$, then:*

$$D^{\alpha} y(x) = g(x, y(x)), \tag{28}$$

is called a FDE of the RL type with initial conditions:

$$D^{\alpha-k} y(0) = h_k, \ (k = 1, 2, 3, ..., n-1). \tag{29}$$

Similarly,

$$D_*^{\alpha} y(x) = g(x, y(x)), \tag{30}$$

is called a fractional differential equation of the Caputo type with initial conditions:

$$D^{k} y(0) = h_k, \ (k = 1, 2, 3, ..., n-1). \tag{31}$$

5 Conclusion

By doing a study of the literature on the definitions and properties of fractional calculus, we are able to recollect some fundamental details and prerequisites in this work. This would give you enough information on this crucial subject. For completeness, several lemmas and theorems are illustrated in detail.

6 Open Problem

From the perspective of the presented theories, one might further investigate and explore certain conditions further results, facts, and theorems related to the field of fractional calculus.

References

[1] Chen, Chaó-Kuang, and Shing-Huei Ho. "Application of differential transformation to eigenvalue problems." Applied mathematics and computation 79.2-3 (1996): 173-188.

[2] Chen, C. L., and Y. C. Liu. "Solution of two-point boundary-value problems using the differential transformation method." Journal of Optimization Theory and Applications 99.1 (1998): 23-35.

[3] Linares, Felipe, Ademir Pastor, and Jean-Claude Saut. "Well-posedness for the ZK equation in a cylinder and on the background of a KdV soliton." Communications in Partial Differential Equations 35.9 (2010): 1674-1689.

[4] Infeld, E., and P. Frycz. "Self-focusing of nonlinear ion-acoustic waves and solitons in magnetized plasmas. Part 2. Numerical simulations, two-soliton collisions." Journal of plasma physics 37.1 (1987): 97-106.

[5] Infeld, E., and George Rowlands. "Stability of nonlinear ion sound waves and solitons in plasmas." Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 366.1727 (1979): 537-554.

[6] Allen, M. A., and G. Rowlands. "Determination of the growth rate for the linearized Zakharov—Kuznetsov equation." Journal of plasma physics 50.3 (1993): 413-424.

[7] Mišutka, Jozef, and Leo Galamboš. "System description: Egomath2 as a tool for mathematical searching on wikipedia. org." International Conference on Intelligent Computer Mathematics. Springer, Berlin, Heidelberg, 2011.

[8] Odibat, Zaid M., and Nabil T. Shawagfeh. "Generalized Taylor's formula." Applied Mathematics and Computation 186.1 (2007): 286-293.

[9] He, Ji-Huan. "Application of homotopy perturbation method to nonlinear wave equations." Chaos, Solitons Fractals 26.3 (2005): 695-700.

[10] Singh, Palwinder, Sanjay K. Srivastava, and Upma Arora. "Stability of SEIR model of infectious diseases with human immunity." Global J. Pure Appl. Math 13.6 (2017): 1811-1819.

[11] Tymoczko, Dmitri. A geometry of music: Harmony and counterpoint in the extended common practice. Oxford University Press, 2010.

[12] Van den Driessche, Pauline. "Reproduction numbers of infectious disease models." Infectious Disease Modelling 2.3 (2017): 288-303.

[13] Van den Driessche, Pauline, and James Watmough. "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission." Mathematical biosciences 180.1-2 (2002): 29-48.

[14] Wald, Ellen R., Kathryn M. Schmit, and Daniele Y. Gusland. "A pediatric infectious disease perspective on COVID-19." Clinical Infectious Diseases 72.9 (2021): 1660-1666.

[15] Wirkus, Stephen A., and Randall J. Swift. A course in ordinary differential equations. CRC press, 2006.

[16] He, Ji-Huan. "Application of homotopy perturbation method to nonlinear wave equations." Chaos, Solitons Fractals 26.3 (2005): 695-700.

[17] Allen, M. A., and G. Rowlands. "Determination of the growth rate for the linearized Zakharov—Kuznetsov equation." Journal of plasma physics 50.3 (1993): 413-424.

[18] Batiha, Iqbal M., et al. "A Numerical Confirmation of a Fractional-Order COVID-19 Model's Efficiency." Symmetry 14.12 (2022): 2583.

[19] Batiha, Iqbal M., et al. "Modified Three-Point Fractional Formulas with Richardson Extrapolation." Mathematics 10.19 (2022): 3489.

[20] He, Ji-Huan. "Homotopy perturbation method for solving boundary value problems." Physics letters A 350.1-2 (2006): 87-88.

[21] Wang, Qi. "Homotopy perturbation method for fractional KdV equation." Applied Mathematics and Computation 190.2 (2007): 1795-1802.

[22] Arshad, Muhammad, Dianchen Lu, and Jun Wang. "(N+ 1)-dimensional fractional reduced differential transform method for fractional order partial differential equations." Communications in Nonlinear Science and Numerical Simulation 48 (2017): 509-519.

[23] Odibat, Zaid M., et al. "A multi-step differential transform method and application to non-chaotic or chaotic systems." Computers & Mathematics with Applications 59.4 (2010): 1462-1472.

[24] Albadarneh, Ramzi B., Iqbal M. Batiha, and Mohammad Zurigat. "Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM)." Int. J. Math. Comput. Sci 16.1 (2016): 103-111.

[25] Batiha, Iqbal M., et al. "Dynamics analysis of fractional-order Hopfield neural networks." International Journal of Biomathematics 13.08 (2020): 2050083.

[26] Batiha, Iqbal M., et al. "Fractional-order coronavirus models with vaccination strategies impacted on Saudi Arabia's infections." AIMS Mathematics 7.7 (2022): 12842-12858.