Int. J. Open Problems Compt. Math., Vol. 16, No. 1, March 2023
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Some Properties of ν-Fuzzy Prime Hyperideals of Γ-Hyperrings

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Received 12 September 2021; Accepted 3 January 2022

Abstract

In this paper, the definition of hypergroups was edited and the notion of η-fuzzy hyperideals of of Γ-hyperrings are introduced and basic properties of them are investigated. In particular, the representation theorem for η-fuzzy hyperideals of R are given. We consider some properties of $\eta-$ fuzzy prime hyperideals of R according to Swamys definition and the complete $\eta-$ fuzzy prime hyperideals.

Keywords: Γ- *hyperring, (ν-fuzzy) hyperideal, fuzzy polygroup, canonical hypergroup, fuzzy product.*

2010 Mathematics Subject Classification: 03F55, 46S40.

1 Introduction

The concept of hyper structures was introduced in 1934 by Marty [19] at the 8^{th} congress of Scandinavian mathematicians. Hyper structures have many applications to several branches of both pure and applied sciences (for example see [1], [2], [4], [5], [8], [9], [10], [12], [13], [20], [25], [28]).

Fuzzy sets were defined by Zadeh in 1965 [26]. The fuzzy set theory were developed by Zadeh himself and many researchers in mathematics. For example the concept of a fuzzy group was introduced by Rosenfeld [22] and the notion of fuzzy ideal in a ring introduced and studied by Liu [18]. Recently fuzzy set theory have been had good develop in hyperstructures theory (see [6], [15], [27]). Corcini et. al. in a recent book on hyperstructures [11] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

The notion of Γ-rings was introduced by Nobosawa [21] and immediately after him in 1966, Barnes [7] extended this notion and obtained more results. Almost 10 years later Kyuno [16, 17] investigated new aspects of Γ-rings such as: prime Γ-rings and left and right unities of Γ-rings. Also in recent years Ozturk et. al. [14] applied the concept of fuzzy sets to the theory of Γ-rings.

In this paper, we show that the first condition of the definition of hypergroups and third condition of the definition of hyperring are meaningless and we redefine them. Then we introduce the notion of (ν-)fuzzy hyperideals of Γ-hyperrings and obtain some related basic results, such as: characterization of (ν-)fuzzy hyperideals based on their level subsets and construction new (ν-fuzzy) hyperideals by given fuzzy hyperideal of Γ-hyperrings. In particular, we show that under certain conditions ν-fuzzy hyperideals of Γ-hyperrings become two-valued. We describe ν-fuzzy hyperideals of the product of Γ-hyperrings. As a beautiful example based on our previous Geometric work [23], we prove that L-Fuzzy tangent spaces of C^∞ L-fuzzy manifolds with gradation of opennes, are hypergroups.

2 Notations and Preliminaries

In this section we gather all definitions and simple properties which we require of semihyperrings and fuzzy subsets notions.

Definition 2.1 *[19]. Let H be a nonempty set. A map $+ : H \times H \longrightarrow P^*(H)$ is called hyperoperation or join operation, where $P^*(H)$ is the set of nonempty subsets of H .*

Definition 2.2 *[19]. A set H together a hyperoperation $+$ is called a canonical hypergroup if the following conditions are satisfied*

(1) $(x + y) + z = x + (y + z) \qquad \forall x, y, z \in H;$

(2) $x + y = y + x;$

(3) *There exist an element $0 \in H$ such that for every $x \in H$, there is only one $x' \in H$ such that*
$0 \in x + x' \cap x' + x$ *(we denote x' by $-x$);*

(4) $\forall x, y, z \in H$, $z \in x + y$ *implies* $x \in z - y$.

Remark 2.3 *First condition of the definition of a canonical hypergroup is meaningless. Because for all $x, y, z \in H$, $x + y$ and $y + z$ are elements of*

$P^*(H)$, so additions $(x+y)+z$ and $x+(y+z)$ are inaccurate. we rewrite this definition more precisely as follows:

Definition 2.4 *A set H together a hyperoperation $+$ and two maps*

$$+_1 : P^*(H)\times H \longrightarrow P^*(H) \qquad \text{and} \qquad +_2 : H\times P^*(H) \longrightarrow P^*(H)$$

$$(A,z) \longrightarrow A+_1 z = \{r+z \mid r \in A\} \qquad \text{and} \qquad (x,A) \longrightarrow x+_2 A = \{x+s \mid s \in A\}$$

is called a redefined canonical hypergroup (briefly r-canonical hypergroup) if satisfy the following conditions:

(i) $(x+y)+_1 z = x+_2 (y+z)$ $\forall x,y,z \in H$,

(ii) $x+y = y+x$;

(iii) There exists an element $0 \in H$ such that for every $x \in H$, there is only one $x' \in H$ such that $0 \in x+x' \cap x'+x$ (we denote x' by $-x$);

(iv) $\forall x,y,z \in H$, $z \in x+y$ *implies* $x \in z-y$.

By $U <_h H$, we mean that U is a subhypergroup of H and it is that the restriction of hyperoperation $+$ is closed on U. We denote the set of all subhypergroup of H, by $SH(H)$.
A *canonical hypergroup* is a commutative polygroup.

Since the tangent spaces of an C^∞- manifold play very important roles in many parts of mathematics, as an useful example of r-canonical hypergroup , we investigate the LG-fuzzy tangent space $LGT_p(X)$ to an C^∞ L-fuzzy manifold of dimension n with gradation of openness which we defined in our previous paper [23].

Definition 2.5 *Let $(X,\mathfrak{T})$ be an C^∞ L-fuzzy manifold of dimension n with gradation of openness and $p \in X$. We define the LG-fuzzy tangent space $LGT_p(X)$ to X at p to be the set of all mappings $Z_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha,\beta \in \mathbb{R}$ and $f,g \in C^\infty(p)$ the two conditions:*

i) $Z_p(\alpha f+\beta g) = \alpha(Z_p f)+\beta(Z_p g)$ *(linearity),*

ii) $Z_p(fg) = (Z_p f)g(p)+f(p)(Z_p g)$ *(leibniz rule),*

with the LG-fuzzy vectore space operations in $LGT_p(X)$ defined by:
$(Z_p+W_p)f = Z_p f+W_p f$
$(\alpha Z_p)f = \alpha(Z_p f)$.
If we define $\eta : LGT_p(X) \to L$, $\eta(f) = A(p)$, where (A,ψ) is a LG-fuzzy coordinate neighborhood of p. Then $(LGT_p(X),\ \eta)$ is a L-fuzzy vector space. Each $Z_p \in LGT_p(X)$ is called an LG-fuzzy tangent vector to X at p.

Example 2.6 *Let $(X, \mathfrak{T})$ be a C^1 L-fuzzy manifold of dimension n with gradation of openness and $p \in X$. Let $LTG_p(X)$ be the L-fuzzy tangent space to X at p. Let*

$$\overline{LTG_p(X)} = \{\ \overline{X_p} = \ \{X_p,\ -X_p\} \mid X_p \in LTG_p(X)\ \}.$$

Then $\left(\overline{LTG_p(X)},\ \oplus\right)$ *is a hypergroup with respect to the hyperoperations*

$$\overline{X_p} \oplus \overline{Y_p} = \ \{\ \overline{X_p + Y_p},\ \overline{X_p - Y_p}\ \}.$$

Proof. *$\forall\ X_p, Y_p, Z_p \in LTG_p(X)$, we have:*

(i) $(\overline{X_p} \oplus \overline{Y_p}) \oplus_1 \overline{Z_p} = \overline{X_p} \oplus_2 (\overline{Y_p} \oplus \overline{Z_p})$. *Since:*
$$(\overline{X_p} \oplus \overline{Y_p}) \oplus_1 \overline{Z_p} = \{\ \overline{X_p + Y_p} \oplus \overline{Z_p},\ \overline{X_p - Y_p} \oplus \overline{Z_p}\ \}$$
$$= \{\ \overline{X_p + Y_p + Z_p},\ \overline{X_p + Y_p - Z_p},\ \overline{X_p - Y_p + Z_p},\ \overline{X_p - Y_p - Z_p}\ \},$$

and $(\overline{X_p} \oplus_2 (\overline{Y_p} \oplus \overline{Z_p}) = \{\ \overline{X_p} \oplus \overline{Y_p + Z_p},\ \overline{X_p} \oplus \overline{Y_p - Z_p}\ \}$
$$= \{\ \overline{X_p + Y_p + Z_p},\ \overline{X_p - (Y_p + Z_p)},\ \overline{X_p + Y_p - Z_p},\ \overline{X_p - (Y_p - Z_p)}\ \}$$

(ii) $\overline{X_p} \oplus \overline{Y_p} = \overline{Y_p} \oplus \overline{X_p}$. *Because we see:*
$$\overline{X_p} \oplus \overline{Y_p} = \ \{\ \overline{X_p + Y_p},\ \overline{X_p - Y_p}\ \}$$
$$= \{\ \{\ (X_p + Y_p),\ -(X_p + Y_p)\},\ \{\ (X_p - Y_p),\ (Y_p - X_p)\ \}\ \}$$
$$= \{\ \{\ (Y_p + X_p),\ -(Y_p + X_p)\},\ \{\ (Y_p - X_p),\ (X_p - Y_p)\ \}\ \}$$
$$= \{\ \overline{Y_p + X_p},\ \overline{Y_p - X_p}\ \}$$
$$= \overline{Y_p} \oplus \overline{X_p};$$

(iii) $\forall\ \overline{X_p} \in \overline{LTG_p(X)}$ *there exists an unique element* $\overline{-X_p} \in \overline{LTG_p(X)}$, *such that*

$\overline{0_p} \in (\overline{X_p} \oplus \overline{-X_p}) \cap (\overline{-X_p} \oplus \overline{X_p})$. *Since we have:*

$$\overline{X_p} \oplus \overline{-X_p} = \ \{\ \overline{X_p + (-X_p)},\ \overline{X_p - (-X_p)}\ \} = \{\ \overline{0_p},\ \overline{2X_p}\ \}$$

$$\overline{-X_p} \oplus \overline{X_p} = \ \{\ \overline{-X_p + (X_p)},\ \overline{-X_p - (X_p)}\ \} = \{\ \overline{0_p},\ \overline{-2X_p}\ \}$$

(iv) $\forall\ X_p,\ Y_p,\ Z_p \in LTG_p(X)\ \ \overline{Z_p} \in \overline{X_p} + \overline{Y_p}$ *implies* $\overline{X_p} \in \overline{Z_p} - \overline{Y_p}$. *Since:*
$$\overline{Z_p} \in \overline{X_p} + \overline{Y_p} \implies \overline{Z_p} = \overline{X_p + Y_p} \ \text{ or } \ \overline{Z_p} = \overline{X_p - Y_p}$$
$$\implies Z_p = X_p + Y_p \ \text{ or } \ Z_p = X_p - Y_p$$
$$\implies X_p = Z_p - Y_p \ \text{ or } \ X_p = Z_p + Y_p$$
$$\implies \overline{X_p} = \overline{Z_p - Y_p} \ \text{ or } \ \overline{X_p} = \overline{Z_p + Y_p}$$
$$\implies \overline{X_p} = \overline{Z_p} - \overline{Y_p}.$$

Definition 2.7 *([15],[25]). An algebraic structure $(R, +, .)$ is called a hyperring if:*

(i) $(R, +)$ *is a canonical hypergroup,*

(ii) $(R, .)$ *is a multiplicative semigroup having zero as a bilaterally absorbing element, i.e.,* $x.0 = 0 = 0.x$,

(iii) The multiplication is distributive with respect to the hyperoperation $+$, *i.e.,* $x.(y + z) = x.y + x.z$ *and* $(y + z).x = y.x + z.x$, $\forall x, y, z \in R$.

Remark 2.8 *Notice that the third condition of the definition of hyperring is meaningless. Because for all* $x, y, z \in H$, $x + y$ *and* $y + z$ *are elements of* $P^*(H)$, *so the multiplication* $x.(y + z)$ *and* $(x + y).z$ *are careless. Therefore we need to redefine exactly hyperrings as follows:*

Definition 2.9 *An algebraic structure* $(R, +, .)$ *is called a redefined hyperring (r-hyperring) if:*

(i) $(R, +)$ *is a r-canonical hypergroup,*

(ii) $(R, .)$ *is a multiplicative semigroup having zero as a bilaterally absorbing element, i.e.,* $x.0 = 0 = 0.x$,

(iii) Two multiplication $._1$ *and* $._2$ *defined by*

$$._1 : H \times P^*(H) \longrightarrow P^*(H) \qquad \text{and} \qquad ._2 : P^*(H) \times H \longrightarrow P^*(H)$$

$$(x, A) \longrightarrow x \;._1 A = \{x.r \mid r \in A\} \qquad \text{and} \qquad (A, x) \longrightarrow A \;._2 x = \{s.x \mid s \in A\}$$

are distributive with respect to the hyperoperation $+$, *i.e.,*
$x \;._1 (y + z) = x.y + x.z$ *and* $(y + z) \;._2 x = y.x + z.x$, $\forall x, y, z \in R$.

Definition 2.10 *[3].*

(i) A nonempty subset I *of a hyperring* R *is called a right (resp. left) hyperideal of* R *if* $r.x \in I$ *(resp.* $x.r \in I$*)* $\forall r \in R$, $\forall x \in I$;

(ii) I *is called a hyperideal if* I *is both left and right hyperideal;*

(iii) A proper hyperideal I *of* R $(I \neq R)$ *is called a prime hyperideal if* $a.b \in I$ *implies that* $a \in I$ *or* $b \in I$. *The set of all prime hyperideals of* R *is called the prime spectruR of* R *and it is denoted by* $Spec(R)$.

Definition 2.11 *[3]. Let* $(R, +)$ *and* $(\Gamma, +)$ *be canonical hypergroups. Then* R *is said to be an* Γ*- hyperring if there exists a mapping* $\cdot : R \times \Gamma \times R \longrightarrow P^*(R)$ *such that the following conditions are satisfied:*

(1) $(x + y)\alpha z \subseteq x\alpha z + y\alpha z$, $x\alpha(y + z) \subseteq x\alpha y + x\alpha z$; $\forall x, y \in R$, $\forall \alpha \in \Gamma$

(2) $x(\alpha+\beta)y \subseteq x\alpha y + x\beta y;\ \forall x,y \in R,\ \forall \alpha,\beta \in \Gamma$

(3) $(x\alpha y)\beta z \subseteq x\alpha(y\beta z);\ \forall x,y \in R,\ \forall \alpha,\beta \in \Gamma.$

If in definition, we replace all inclusion with equality, then R is called a *strong* Γ*- hyperring.*

Example 2.12 *Let* $(R,+,\cdot)$ *be a arbitrary hyperring and* $(\Gamma,+)$ *be a subhypergroup of* R*, then by*

$$\begin{array}{ccc} \cdot & : & R\times\Gamma\times R \longrightarrow P^*(R) \\ (a,\gamma,b) & \longmapsto & \{z \mid z \in a\gamma b\} \end{array}$$

R *is a* Γ*- hyperring. Since*

(i) $a(\gamma+\gamma')b = \{z\,|z \in a(\gamma+\gamma')b\} = \{z\,|z \in (a\gamma + a\gamma')b\}$
$= \{z\,|z \in a\gamma b + a\gamma' b\} = a\gamma b + a\gamma' b.$

(ii) $a\gamma(b+b') = \{z\,|z \in a\gamma(b+b')\} = \{z\,|z \in a\gamma b + a\gamma b'\} = a\gamma b + a\gamma b'$

(iii) $(a\gamma b)\gamma' b' = \{z\,|z \in (a\gamma b)\gamma' b'\} = \{z\,|z \in a\gamma(b\gamma' b')\}.$

Definition 2.13 *Let* $(R,+)$ *and* $(\Gamma,+)$ *be r-canonical hypergroups. Then* R *is called a redefined* Γ*- hyperring if there exists* $\cdot : R\times\Gamma\times R \longrightarrow P^*(R)$ *and three relative maps:*

$\cdot_1 : P^*(R)\times\Gamma\times R \longrightarrow P^*(R)$ *and* $\cdot_2 : R\times\Gamma\times P^*(R) \longrightarrow P^*(R)$

$(A,\alpha,z) \longrightarrow A._1\alpha._1 z = \{r\alpha z \mid r \in A\},\quad (x,\alpha,A) \longrightarrow x._2\alpha._2 A = \{x\alpha s \mid s \in A\}$

$$\cdot_3 : R\times P^*(\Gamma)\times R \longrightarrow P^*(R)$$
$$(x,B,z) \longrightarrow x._3 B._3 z = \{xrz \mid r \in B\}$$

such that the following conditions are satisfied:

(i) $(x+y)._1\alpha._1 z \subseteq x\alpha z + y\alpha z$, $x._2\alpha._2(y+z) \subseteq x\alpha y + x\alpha z;\ \forall x,y,z \in R,$ $\forall \alpha \in \Gamma$

(ii) $x._3(\alpha+\beta)._3 y \subseteq x\alpha y + x\beta y;\ \forall x,y \in R,\ \forall \alpha,\beta \in \Gamma$

(iii) $(x\alpha y)._1\beta._1 z \subseteq x._2\alpha._2(y\beta z);\ \forall x,y \in R,\ \forall \alpha,\beta \in \Gamma.$

Example 2.14 *Let* M , Γ *be two commutative polygroup and* $M = End(\Gamma)$*. Then by following map* M *is a* Γ*- hyperring.*

$$\cdot : M\times\Gamma\times M \longrightarrow P^*(M)$$
$$(f,a,g) \longmapsto \{h \mid h \in f(a)\cdot g\}$$

for all $f,g,h \in M$, $a \in A$ *Since*

(i) $f(a+a')\cdot g = \{z\,|z\in f(a+a')\cdot g\} = \{z\mid z\in (f(a)\cdot g + f(a')\cdot g)\}$
$= f(a)\cdot g + f(a')\cdot g$

(ii) $f(a)\cdot(g+g') = \{T\,|T\in (f(a)\cdot g + f(a)\cdot g')\} = f(a)\cdot g + f(a)\cdot g'$

(iii) $(\,f(a)\cdot(f'(a')\cdot g)) = \{W\,|W\in f(a)\cdot(f'(a')\cdot g)\}$
$= \{W\,|W\in (f(a)\cdot f')(a')\cdot g)\}$
$= (f(a)\cdot f')(a')\cdot g).$

Definition 2.15 *[3]. A right (resp. left) hyperideal of Γ- hyperring R is a subpolygroup U of R such that $U\Gamma R\subseteq U$ $(R\Gamma U\subseteq U)$. Also if $\triangle$ is a subpolygroup of Γ, then the subpolygroup I of R is said to be a right (left) $\triangle$-hyperideal if $I\triangle R\subseteq I$ (resp. $R\triangle I\subseteq I$).*

Clearly every hyperideal of a Γ- hyperring is a $\triangle$-hyperideal for some $\triangle\subseteq\Gamma$. We use $I=[0,1]$, the real unit interval as chain with the usual ordering, in which $\wedge$ stands for minimum or infimum (inf)(or intersection) and $\vee$ stand for maximum or supremum (sup)(or union), for the degree of membership. A fuzzy subset of a given set X is a mapping $\mu : X\to I$. We denote the set of all fuzzy subset of X by $FS(X)$, that is $FS(X)=\{\mu|\mu : X\to[0,1]\}$ is a function.

Definition 2.16 *[25] Let $(R,+)$ be canonical hypergroup and $\mu\in FS(R)$. Then μ is a fuzzy subpolygroup of R, if for all $a,b\in R$ the following conditions hold:*

(1) $\bigwedge\limits_{z\in a+b}\mu(z)\geq\mu(a)\wedge\mu(b)$;

(2) $\mu(-a)\geq\mu(a)$.

We denote the set of all fuzzy subpolygroups of R, by $FP(R)$. In the sequel by R we mean a Γ- hyperring.

Definition 2.17 *[3] (1) A fuzzy subset μ of R is said to be left (resp. right) fuzzy hyperideal of R iff for all $x,y\in R$ and $\gamma\in\Gamma$ we have*

(1-1) $\mu\in FP(R)$,

(1-2) $\bigwedge\limits_{z\in x\gamma y}\mu(z)\geq\mu(y)$ *(resp.* $\bigwedge\limits_{z\in x\gamma y}\mu(z)\geq\mu(x)$).

We denote the set of all fuzzy hyperideals of R by $FHI(R)$.
(2) A fuzzy subset μ of R is said to be a left (resp. right) $\nu-$ fuzzy hyperideal of R iff for all $x,y,z\in R$ and $\gamma\in\Gamma$

(2-1) $\mu\in FP(M)$ *and* $\nu\in FP(\Gamma)$;

(2-2) $\bigwedge\limits_{z\in x\gamma y}\mu(z)\geq\mu(y)\wedge\nu(\gamma)$ *(resp.* $\bigwedge\limits_{z\in x\gamma y}\mu(z)\geq\mu(x)\wedge\nu(\gamma)$).

Clearly every fuzzy hyperideal is $\nu-$ fuzzy hyperideal, for some $\nu\in FP(\Gamma)$.

3 Main Results

First we introduce several binary operation in $FS(R)$.

Definition 3.1 *Let* $\mu, \nu \in FS(R)$. *Then*

(i) $(\mu+\nu)(x) = \bigvee \{\mu(y) \wedge \nu(z) \mid x \in y+z,\ y,z \in R\}$

(ii) $(\mu-\nu)(x) = \bigvee \{\mu(y) \wedge \nu(z) \mid x \in y-z,\ y,z \in R\}$

(iii) $$(\mu\nu)(x) = \bigvee \left\{ \begin{array}{c} \bigwedge\{\mu(y_1),\mu(y_2),...,\mu(y_n),\nu(z_1),\nu(z_2),...,\nu(z_n)\} \\ \Big|\ x \in \sum_{i=1}^{n} y_i\gamma_i z_i,\ n \in \mathbb{N},\ y_i, z_i \in R,\ \gamma_i \in \Gamma \end{array} \right\}$$

(iv) $(\mu \circ \nu)(x) = \bigvee \{\mu(y) \wedge \nu(z) \mid x \in y\gamma z,\ y,z \in R,\ \gamma \in \Gamma\}$

Proposition 3.2 . *If* $\mu \in \nu_1 - FHI(R)$ *and* $\xi \in \nu_2 - FHI(R)$, *then*

$$(\mu+\xi)(z) = \bigvee \{\mu(x) \wedge \xi(y) \mid z \in x+y,\ x,y \in R\}$$

is a $\nu_1 \cap \nu_2$*–fuzzy hyperideal of R. Moreover* $\mu+\xi = \langle \mu \cup \xi \rangle$.

Proof. Let $z \in x+y$ then,

$$\begin{aligned}(\mu+\xi)(z) &= \bigvee \{\mu(u) \wedge \xi(v) \mid z \in x+y\ \} \\ &\geq \bigvee \{\mu(t_1) \wedge \xi(t_2) \mid \exists t_1 \in u_1+v_1, \exists t_2 \in u_2+v_2,\ z \in t_1+t_2, \\ &\qquad \text{x}\in u_1+u_2, y \in v_1+v_2 \\ &\geq \bigvee \{\mu(u_1) \wedge \mu(v_1)) \wedge (\xi(u_2) \wedge \xi(v_2) \mid x \in u_1+u_2,\ y \in v_1+v_2\} \\ &= \{(\mu(u_1) \wedge \xi(u_2)) \wedge (\mu(v_1) \wedge \xi(v_2)) \mid x \in u_1+u_2,\ y \in v_1+v_2\} \\ &= (\bigvee \{\mu(u_1) \wedge \xi(u_2) \mid x \in u_1+u_2\} \wedge (\bigvee \{\mu(v_1) \wedge \xi(v_2) \mid y \in v_1+v_2\}) \\ &= (\mu+\xi)(x) \wedge (\mu+\xi)(y).\end{aligned}$$

$$\begin{aligned}(\mu+\xi)(z) &= \bigvee \{\mu(u) \wedge \xi(v) \mid z \in u+v\ \} \\ &\geq \bigvee \{\mu(-u) \wedge \xi(-v) \mid -z \in -u-v\ \} \\ &= (\mu+\xi)(-z)\end{aligned}$$

Let $z \in x\gamma y$ then we have

$$\begin{aligned}(\mu+\xi)(z) &= \bigvee \{\mu(a) \wedge \xi(b) \mid z \in a+b\ \} \\ &\geq \bigvee \{\mu(t_0) \wedge \xi(s_0) \mid \exists t_0 \in x\gamma y, \exists s_0 \in u\gamma v,\ z \in t_0+s_0,\ y \in u+v\} \\ &\geq \bigvee \{\mu(u) \wedge \eta_1(\gamma) \wedge \xi(v) \wedge \eta_2(\gamma) \mid y \in u+v\ \} \\ &= (\mu+\xi)(y) \wedge (\eta_1 \cap \eta_2)(\gamma).\end{aligned}$$

similarly $(\mu+\xi)(z) \geq (\mu+\xi)(x) \wedge (\eta_1 \cap \eta_2)(\gamma)$. Then we have:
$(\mu+\xi)(z) \geq [(\mu+\xi)(x) \vee (\mu+\xi)(y)] \wedge (\eta_1 \cap \eta_2)(\gamma)$.

since $\mu(0) = \xi(0) \Rightarrow \mu, \xi \subseteq \mu+\xi \Rightarrow \mu \cup \xi \subseteq \mu+\xi \Rightarrow \mu+\xi \subseteq \langle \mu \cup \xi \rangle$.
Now it is enough that we prove if $\zeta \in \eta_1 \cap \eta_2$ and $\mu \cup \xi \subseteq \zeta$, then

$$\begin{aligned}(\mu+\xi)(z) &= \bigvee \{\mu(x) \wedge \xi(y) \mid z \in x+y\ \} \\ &\leq \bigvee \{\zeta(x) \wedge \zeta(y) \mid z \in x+y\ \} \\ &\leq \zeta(z).\end{aligned}$$

therefore $\mu+\xi \subseteq \zeta \Rightarrow \mu+\xi = \langle \mu \cup \xi \rangle$.

Proposition 3.3 *If μ is a $\eta-$fuzzy hyperideal of Γ- hyperring R and $\nu \in FS(R)$, then*

$$\mu\nu(z) = \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(x_i) \wedge \nu(y_i)) \,\middle|\, z \in \sum_{i=1}^{n} x_i\gamma_i y_i,\ n \in N,\ x_i, y_i \in R,\ \gamma_i \in \Gamma \right\}$$

is a $\eta-$fuzzy hyperideal of R. Moreover $\mu\nu = \langle \mu \circ \nu \rangle$.

Proof. Let $z \in x + y$, then we have:

$$\begin{aligned}
&\mu\nu(z) = \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(a_i) \wedge \nu(b_i)) \,\middle|\, z \in \sum_{i=1}^{n} a_i\gamma_i b_i,\ n \in N,\ a_i, b_i \in R,\ \gamma_i \in \Gamma \right\} \\
&\geq \bigvee \left\{ \bigwedge_{j=1}^{m} (\mu(x_j) \wedge \nu(y_j)) \wedge \bigwedge_{k=1}^{l} (\mu(x_k^{'}) \wedge \nu(y_k^{'})) \,\middle|\, x \in \sum_{j=1}^{m} x_j\gamma_j y_j,\ \ y \in \sum_{k=1}^{l} x_k^{'}\gamma_k^{'} y_k^{'}, \right\} \\
&= \bigvee \left\{ \bigwedge_{j=1}^{m} (\mu(x_j) \wedge \nu(y_j)) \,\middle|\, x \in \sum_{j=1}^{m} x_j\gamma_j y_j,\ x_j, y_j \in R,\ \ \gamma_j \in \Gamma \right\} \\
&\wedge \bigvee \left\{ \bigwedge_{k=1}^{l} (\mu(x_k^{'}) \wedge \nu(y_k^{'})) \,\middle|\, y \in \sum_{k=1}^{l} x_k^{'}\gamma_k^{'} y_k^{'},\ l \in N,\ x_k^{'},\ y_k^{'} \in R,\ \gamma_k^{'} \in \Gamma \right\} \\
&= (\mu\nu)(x) \wedge (\mu\nu)(y).
\end{aligned}$$

$$\begin{aligned}
\mu\nu(z) &= \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(x_i) \wedge \nu(y_i)) \,\middle|\, z \in \sum_{i=1}^{n} x_i\gamma_i y_i,\ n \in N,\ x_i, y_i \in R,\ \gamma_i \in \Gamma \right\} \\
&\geq \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(-x_i) \wedge \nu(-y_i)) \,\middle|\, -z \in \sum_{i=1}^{n} (-x_i)(-\gamma_i)(-y_i),\ n \in N,\ \ -x_i, -y_i \in R, \right\} \\
&= \mu\nu(-z)
\end{aligned}$$

Now let $z \in x\gamma y$, then we have:

$$\begin{aligned}
\mu\nu(z) &= \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(a_i) \wedge \nu(b_i)) \,\middle|\, z \in \sum_{i=1}^{n} a_i\gamma_i b_i,\ n \in N,\ a_i, b_i \in R,\ \gamma_i \in \Gamma \right\} \\
&\geq \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(t_i) \wedge \nu(v_i)) \,\middle|\, \exists t_i \in x\gamma v_i, y \in \sum_{i=1}^{n} u_i\gamma_i v_i,\ n \in N,\ u_i, v_i \in R, \right\} \\
&\geq \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(u_i) \wedge \eta(\gamma) \wedge \nu(v_i)) \,\middle|\, y \in \sum_{i=1}^{n} u_i\gamma_i v_i,\ n \in N,\ u_i, v_i \in R, \right\} \\
&= \mu\nu(y) \wedge \eta(\gamma).
\end{aligned}$$

similarly, $(\mu\nu)(z) \geq (\mu\nu)(x) \wedge \eta(\gamma)$. Therefore $(\mu\nu)(z) \geq [(\mu\nu)(x) \vee (\mu\nu)(y)] \wedge \eta(\gamma)$.

Clearly $\mu \circ \nu \subseteq \mu\nu$. Hence $\langle \mu \circ \nu \rangle \subseteq \mu\nu$.

Now it is enough we prove that if $\zeta \in \eta - FHI(R)$ and $\mu \circ \nu \subseteq \zeta$ then $\mu\nu \subseteq \zeta$.

$$\begin{aligned}
\mu\nu(z) &= \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(x_i) \wedge \nu(y_i)) \,\middle|\, z \in \sum_{i=1}^{n} x_i\gamma_i y_i,\ n \in N,\ x_i, y_i \in R,\ \gamma_i \in \Gamma \right\} \\
&\leq \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu \circ \nu)(t_i) \,\middle|\, z \in \sum_{i=1}^{n} x_i\gamma_i y_i,\ n \in N,\ \exists t_i \in x_i\gamma_i y_i,\ x_i, y_i \in R,\ \gamma_i \in \Gamma \right\}
\end{aligned}$$

$\leq \bigvee\left\{ \bigwedge_{i=1}^{n} \zeta(t_i) \middle| z \in \sum_{i=1}^{n} x_i\gamma_i y_i,\ n \in N,\ \exists t_i \in x_i\gamma_i y_i,\ x_i, y_i \in R,\ \gamma_i \in \Gamma \right\}$

$\bigvee\left\{ \zeta(t_{i1}) \wedge \zeta(t_{i2}) \wedge ... \wedge \zeta(t_{in}) \middle| z \in \sum_{j=1}^{n} (t_{ij}),\ n \in N,\ \exists (t_{ij}) \in x_j\gamma_j y_j, x_j, y_j \in R \right\}$

$\leq \bigvee\left\{ \zeta(t_{ij}) \middle| z \in \sum_{j=1}^{n} (t_{ij}),\ n \in N,\ \exists (t_{ij}) \in x_j\gamma_j y_j,\ x_j, y_j \in R, \gamma_j \in \Gamma \right\}$

$= \zeta(z)$.

Therefore $\mu\nu = \langle \mu \circ \nu \rangle$.

Proposition 3.4 *Let* $\mu, \nu, \zeta \in FS(R)$ *as* Γ*- hyperring. Then*

$$\mu \circ (\nu + \zeta) \subseteq \mu \circ \nu + \mu \circ \zeta.$$

Proof. Let $w, u, v \in R$ and $w \in u\gamma v$. Then we have

$\mu \circ (\nu + \zeta)(w) = \bigvee \{\mu(u) \wedge (\nu + \zeta)(v) \mid w \in u\gamma v,\ u, v \in R,\ \gamma \in \Gamma\}$

$= \mu(u) \wedge (\bigvee \{\nu(y) \wedge \zeta(z)) \mid v \in y + z \})$

$= \bigvee \{(\mu(u) \wedge \nu(y)) \wedge (\mu(u) \wedge \zeta(z)) \mid v \in y + z\}$

$\leq \bigvee \{(\mu(u) \wedge \nu(y)) \wedge (\mu(u) \wedge \zeta()z) \mid u\gamma v \subseteq u\gamma y + u\gamma z \}$

$\leq \bigvee\left\{ [\bigwedge_{t_1 \in u\gamma y} (\mu \circ \nu(t_1))] \wedge [\bigwedge_{t_2 \in u\gamma z} (\mu \circ \zeta)(t_2)] \mid u\gamma v \subseteq u\gamma y + u\gamma z \right\}$

$\leq \bigvee \{[(\mu \circ \nu)(t_3) \wedge (\mu \circ \zeta)(t_4)] \mid \exists t_3 \in u\gamma y,\ \exists t_4 \in u\gamma z,\ w \in t_3 + t_4,\ u\gamma v \subseteq u\gamma y + u\gamma z\}$

$\leq (\mu \circ \nu + \mu \circ \zeta)(w)$.

Thus $\mu \circ (\nu + \zeta) \subseteq \mu \circ \nu + \mu \circ \zeta$.

Theorem 3.5 *[3] Let* a_t *be a fuzzy point of* Γ*- hyperring* R*. The fuzzy set* μ *is defined by*

$$\mu(r) = \begin{cases} t & if\ \ r \in (a) \\ 0 & if\ \ r \notin (a) \end{cases}$$

Then $\mu = \langle a_t \rangle$.

Proof. First we show that $\mu \in \eta$-FHI(R). Because the level sub set of μ , that is $\mu_t = (a)$ and $\mu_0 = R$ are fuzzy hyperideals of R.

Therefore $\mu \in \eta$-FHI(R). Also since $\mu(a) = t$ thus $a_t \in \mu$, that is $a_t \subseteq \mu$.

Now we suppose that $\nu \in \eta$-FHI(R) and $a_t \in \nu$. We show that $\mu \subseteq \nu$. Suppose $r \in R$. If $mu(r) = 0$ then $\nu(r) \geq 0 = \mu(r)$.

Other wise $\mu(r) = \mu$ that is $r \in (a)$ hence we have $\nu(r) \geq \nu(a)$.

On the other hand by suppose we have $\nu(a) \geq t$. Hence $\nu(r) \geq t = \mu(r)$. So $\nu \supseteq \mu$ therefore $\mu = \langle a_t \rangle$.

Lemma 3.6 *Let* $\mu, \nu \in FS(R)$*. The* $\mu \subseteq \nu$ *iff* $x_t \in \mu$ *concludes that* $x_t \in \nu$*, for all* $x_t \in R$.

Proof. Lemma 0.5, page 4, of [14] .

Example 3.7 (1) *If* I *be a hyperideal of* R *then* $\chi_I \in FHI(R)$.
(2) *Let* R *be a commutative polygroup. Then*

$$\mu(x) = \begin{cases} 1 & if\ x = 0 \\ \alpha & if\ x \neq 0, \quad 0 \leq \alpha \leq 1 \end{cases}$$

is a fuzzy hyperideal of R.

(3) *Let* $\{u_i\}$ *is a descending chain of hyperideals of* R . *Then fuzzy set* μ *defined by:*

$$\mu(x) = \begin{cases} \frac{n}{n+1} & if \quad x \in u_n \setminus u_{n+1} \\ 1 & if \quad x \in \bigcap\limits_{n=0}^{\infty} u_n \end{cases}$$

is a left fuzzy hyperideal of R.
(4) *Let* $\{u_i\}$ *is a strongly ascending chain of hyperideals of* R. *Then fuzzy subset* μ *defined by:*

$$\mu(x) = \begin{cases} 0 & if \quad x \notin u_i \\ \frac{1}{k} & if \quad k = \wedge \{i \in \mathbb{N} \mid x \in u_i\} \end{cases}$$

is a fuzzy hyperideal of R.

4 $\eta-$ Fuzzy Prime Hyperideal

Theorem 4.1 *Let* R *be a* Γ*-hyperring and* $\mu \in F(R)$. *Then the following conditions are equivalent:*

(i) μ *is a* η*- fuzzy hyperideal of* R.

(ii) a) μ *by operation* $+$ *is a subgroup of* R.

b) $\chi_M \circ \mu \leq \mu$, $\mu \circ \chi_M \leq \mu$

(iii) $\mu - \mu \leq \mu$ *and for all* $r \in R$, $(r\mu \vee \mu r) \leq \mu$

Proof. $(i) \to (ii)$ Let $\mu \in \eta - FHI(R)$. So first $\mu(-x) \geq \mu(x)$, second by hypothesis, we have:

$$\mu(z) \geq \{\mu(x) \wedge \mu(y) \mid z \in x + y, \quad x, y \in R\} \tag{1}$$

Therefore

$$\begin{aligned}(\mu+\mu)(x) &= \bigvee_{x\in r+s\ ,r,s\in R} \{\mu(r)\wedge\mu(s)\} \\ &\leq \bigvee \{\mu(t)\,|\, t\in r+s,\ r,s\in R\} \qquad \text{by (4.1)} \\ &= \bigvee_{x\in r+s} \mu(x)=\mu(x)\end{aligned}$$

Now we show hat $\chi_M\circ\mu\subseteq\mu$ and $\mu\circ\chi_M\leq\mu$ But by suppose we have

$$\mu(z)\geq\{\mu(x)\wedge\ \mu(y)\,|\, z\in x\gamma y,\quad x,y\in R\} \tag{2}$$

Now we suppose $x\in R$. Thus

$$\begin{aligned}\mu\circ\chi_M(x) &= \bigvee_{x\in r\gamma s} (\mu(r)\wedge\chi_M(s)) \\ &\leq \bigvee \{\mu(x)\wedge 1\,|\, x\in r\gamma s,\quad r,s\in R,\ \gamma\in\Gamma\} \qquad \text{by (2)} \\ &= \bigvee_{x\in r\gamma s} \{\mu(x)\}=\mu(x).\end{aligned}$$

We can similarly show that $\chi_M\ \circ\mu\leq\mu$.

$(ii)\to(i)$ Let $\mu\in FS(R)$ satisfy in (a) and (b). Since μ is a fuzzy subgroup of R under '+', hence

$$\mu+\mu\leq\mu,\quad \mu(-x)\geq\mu(x) \qquad \text{for all } x\in R \tag{3}$$

Let $z\in x-y$ therefore for all $x,y\in R$ we have:

$$\begin{aligned}\mu(z) &\geq (\mu+\mu)(z) \\ &= \bigvee_{z\in r+s,\ r,s\in R} \{\mu(r)\wedge\mu(s)\} \\ &= \{\mu(x)\wedge\mu(-y)\}\end{aligned}$$

Now we prove that for all $x,y\in R$,

$$\mu(z)\geq\bigvee\{\mu(x),\mu(y)\,|z\in x\gamma y,\quad z\in R,\ \gamma\in\Gamma\}$$

Since we have: $\mu\geq\ \chi_M\circ\mu$, so for all $z\in x\gamma y$ we have:

$$\begin{aligned}\mu(z) &\geq (\ \chi_M\circ\mu)(z) \\ &= \bigvee_{r,s\in R} \{\chi_M(r)\wedge\mu(s)\} \\ &\geq \{\chi_M(x)\wedge\mu(y)\} \\ &= 1\wedge\mu(y)=\mu(y).\end{aligned} \tag{4}$$

And similarly since $\mu \geq \mu \circ \chi_M$ we have:

$$\mu(z) \geq \mu(x) \tag{5}$$

Therefore from (4.5),(4.6) we conclude that

$$\mu(z) \geq \bigvee \{\mu(x), \mu(y) \,|z \in x\gamma y,\ x, y, z \in R,\ \gamma \in \Gamma\}\,.$$

$(i) \to (iii)$ Let $\mu \in \eta - FHI(R)$. Then for all $x \in R$ we have:

$$\begin{aligned} (\mu - \mu)(x) &= \bigvee_{x \in y - z} \{\mu(y) \wedge \mu(z)\} \\ &\leq \bigvee_{x \in y - z} \mu(x) = \mu(x). \end{aligned}$$

Also for all $x, r, y, y' \in R$

$$\begin{aligned} (r\mu \vee \mu r)(x) &= (r\mu)(x) \vee (\mu r)(x) \\ &\leq \bigvee(\bigvee_{x \in r\gamma y} (\mu(r), \mu(y)), \bigvee_{x \in y'\gamma r} (\mu(y'), \mu(r))) \\ &\leq \bigvee_{x \in r\gamma y} (\mu(x), \mu(x)) \\ &= (\mu(x) \vee \mu(x)) = \mu(x). \end{aligned}$$

$(iii) \to (i)$ Let $\mu - \mu \subseteq \mu$ and $\bigvee(r\mu, \mu r) \subseteq \mu$. So for all $x, y \in R$ and $z \in x - y$

$$\begin{aligned} \mu(z) &\geq (\mu - \mu)(z) \\ &= \bigvee\{\mu(t) \wedge \mu(k)|\ z \in t - k, \quad t, k \in R\} \\ &\geq \{\mu(x) \wedge \mu(y)\} \end{aligned} \tag{6}$$

Since for all $r \in R$, we have $\bigvee(\mu r, r\mu) \subseteq \mu$. Hence $r\mu \subseteq \mu$, and $\mu r \subseteq \mu$. Now if we set $r = x$, then

$$\begin{aligned} \mu(z) &\geq \bigvee_{x\gamma t = x\gamma y} \{(x\mu)(z) \,|z \in x\gamma y \quad, x, y, z \in R\ ,\ \gamma \in \Gamma\} \\ &\geq \mu(y) \end{aligned} \tag{7}$$

similarly by putting $r = y$ we have:

$$\mu(z) \geq \mu(x), \quad \text{where } z \in\ x\gamma y. \tag{8}$$

Therefore from (7),(8) we conclude that

$$\mu(z) \geq \bigvee \{\mu(x), \mu(y) \,|\ \ z \in\ x\gamma y, x, y, z \in R\ ,\ \gamma \in \Gamma\} \tag{9}$$

Therefore we have $\mu \in \eta - FHI(R)$.

Definition 4.2 *[24] Let* $p \in \eta - FHI(R)$ *be non-constant.* p *is called a* $\eta - fuzzy\ prime\ hyperideal$ *if and only if for any* $\mu, \nu \in \eta - FHI(R)$

$$\mu\nu \subseteq p \implies (\mu \subseteq p \quad or \quad \nu \subseteq p)$$

Remark 4.3 *we can say that* $p \in \eta - FHI(R)$ *is prime if and only if* p *be non-constant and for all* $\mu, \nu \in \eta - FHI(R)$.

$$\mu \circ \nu \subseteq p \implies \mu \subseteq p \quad or \quad \nu \subseteq p$$

Definition 4.4 *Let* $p \in \eta - FHI(R)$ *and* p *be non constant . We say that* p *is complete* $\eta - fuzzy\ prime\ hyperideal$ *of R if and only if for any fuzzy point* a_t *and* b_s *of* $\Gamma-$*hyperring R such that* $a, b \in R$ *and* $s, t \in [0, 1]$ *we have:*

$$a_t \circ b_s \subseteq p \implies (a_t \subseteq p \quad or \quad b_s \subseteq p).$$

Lemma 4.5 *Let* a_t *and* b_s *(For all* $a, b \in R$ *and* $s, t \in [0, 1]$*) are two arbitrary fuzzy points of* $\Gamma-$*hyperring R. Then*

$$a_t \circ b_s = (a\gamma b)_{(t\wedge s)}$$

Proof. Let $x \in R$

$$\begin{aligned}(a_t \circ b_s)(x) &= \bigvee \{a_t(y) \wedge b_s(z)\, | x \in y\gamma z\ ,\ y, z \in R\ ,\ \gamma \in \Gamma\} \\ &= \begin{cases} (t \wedge s) & if \quad x \in a\gamma b, \\ 0 & if \quad otherwise \end{cases} \\ &= (a\gamma b)_{(t\wedge s)}(x)\end{aligned}$$

Lemma 4.6 *If* R *be a commutative* $\Gamma-$*hyperring . Then for all fuzzy points* a_t *and* b_s *of* R *we have:*

$$\langle a_t \rangle \circ \langle b_s \rangle \subseteq \langle a_t \circ b_s \rangle$$

Proof. For every $x \in R$

$$\langle a_t \circ b_s \rangle (x) = \left\langle (ab)_{(t\wedge s)} \right\rangle (x) = \begin{cases} (t \wedge s) & if \quad x \in (a\gamma b), \\ 0 & if \quad otherwise \end{cases}$$

$$\begin{aligned}\langle a_t \rangle \circ \langle b_s \rangle (x) &= \bigvee \{\langle a_t \rangle (y) \wedge \langle b_s \rangle (z)\, | x \in y\gamma z\ ,\ y, z \in R\ ,\ \gamma \in \Gamma\} \\ &= \begin{cases} (t \wedge s) & if \quad z \in (b),\ y \in (a),\ x \in y\gamma z \\ 0 & if \quad otherwise \end{cases}\end{aligned}$$

Then by theorems universal algebra, for $z \in (b),\ y \in (a)$, we have $x \in y\gamma z \in (a)(b) \subseteq (a\gamma b)$. Thus

$$\langle a_t \circ b_s \rangle (x) = t \wedge s$$

Theorem 4.7 *(i) Every complete $\eta-$ fuzzy prime hyperideal of R is a $\eta-$ fuzzy prime hyperideal of R.*

(ii) Conversely, if R be a commutative, then every $\eta-$ fuzzy prime hyperideal of R is a complete $\eta-$ fuzzy prime hyperideal of R.

Proof. (i) Let p is a complete $\eta-$ fuzzy prime hyperideal of R and ν, k are two arbitrary $\eta-$ fuzzy prime hyperideal of R. So $\nu k \subseteq p$. we will show that $k \subseteq p$ or $\nu \subseteq p$.
Assume that $\nu\ nsubseteqp$ and $k\ n \subseteq p$. Hence there exist $a \in R$ such that $\nu(a) \leq p(a)$. Therefore $a_{\nu(a)} \notin p$. Now assume that $b \in R$. Then about $a_{\nu(a)}$ and b we can say $a_{\nu(a)} \in \nu,\ b_{k(b)} \in k$. Thus for $r \in R$

$$a_{\nu(a)} \circ\ b_{k(b)}(r) = (a\gamma b)_{\ (\nu(a)\wedge k(b))}\ (r) \qquad \text{by lemma 4.5}$$

$$= \begin{cases} (\nu(a) \wedge\ k(b)) & if \quad r \in na\gamma nb \\ 0 & if \quad otherwise \end{cases}$$

$$\leq\ \bigwedge \Big\{ \nu(a_1), ...,\ \nu(a_n),\ k(b_1), ...,\ k(b_n),\quad r \in na\gamma nb = \sum a_i\gamma_i\, b_i \Big\}$$

$$= \nu k(r) \leq p(r).$$

Therefore $a_{\nu(a)} \circ b_{k(b)} \subseteq p$.

Since p is complete $\eta-$ fuzzy prime hyperideal of R and $a_{\nu(a)} \notin p$ we conclude that $b_{k(b)} \in p$. That is $k(b) \leq p(b)$. Since b was a arbitrary member of R so we have $k \subseteq p$.

(ii) Let R is a commutative $\Gamma-$hyperring and p be a $\eta-$ fuzzy prime hyperideal of R. We prove that p is complete $\eta-$ fuzzy prime hyperideal of R. Therefore we assume a_t, b_s are fuzzy points of R. Hence $a_t \circ b_s \subseteq p$ So we have:

$$\langle a_t \rangle \circ \langle b_s \rangle \subseteq \langle a_t \circ b_s \rangle \subseteq p \qquad \text{by lemma 4.6}$$

But since p is prime hence

$$\langle a_t \rangle \circ \langle b_s \rangle \subseteq p \implies (\langle a_t \rangle \subseteq p \quad \text{or} \quad \langle b_s \rangle \subseteq p) \implies (a_t \in p \quad \text{or} \quad b_s \in p)$$

That is p is a complete prime $\eta-$ fuzzy prime hyperideal of R.

Theorem 4.8 *Let a_t, b_s are fuzzy points of T_pR. Then*

(i) $$\chi_M a_t\ (r) = \begin{cases} t & if \quad r \in p\gamma a \\ 0 & if \quad otherewise, \end{cases}$$

(ii) $$a_t \chi_M\ (r) = \begin{cases} t & if \quad r \in a\gamma q \quad for \quad q \in T_pR \\ 0 & if \quad otherewise, \end{cases}$$

(iii) $(a_t\chi_{_M})b_s(r) = \begin{cases} (t \wedge s) & if \quad r \in z\gamma b, \quad z \in a\ \gamma u,\ u \in R \\ 0 & if \quad otherewise, \end{cases}$

(iv) $(\chi_{_M} a_t)\chi_{_M}(r) = \begin{cases} t \quad if & r \in \sum_{i=1}^{n} z\gamma_i q_i, \quad z \in \sum_{i=1}^{n} u_i\gamma_i a,\ q_i,\ u_i \in T_pR\ ,\ n \in \mathbb{N}, \\ 0 \quad if & otherewise \end{cases}$

Proof.

(i) For all $r \in T_pR$ we have:

$$\begin{aligned} \chi_{_M} a_t(r) &= \begin{cases} \bigvee\{\chi_{_M}(p_1) \wedge ... \wedge\ \chi_{_M}(p_n) \wedge\ a_t(q_1) \wedge ... \wedge\ a_t(q_n)\}\ if\ r \in \sum_{i=1}^{n} p_i\gamma_i q_i\ , \\ 0 \qquad if \quad otherwise, \end{cases} \\ &= \begin{cases} (1 \wedge t) \quad if \quad r \in \sum_{i=1}^{n} p_i\gamma_i a\ , \quad p_i \in T_pR, \\ 0 \qquad if \quad otherwise, \end{cases} \\ &= \begin{cases} t \quad if \quad r \in \sum_{i=1}^{n} p_i\gamma_i a\ , \quad p_i \in T_pR, \quad \gamma_i \in \Gamma, \\ 0 \quad if \quad otherwise. \end{cases} \end{aligned}$$

(ii) This can be proved similar to (i).

(iii) For each $r \in T_pR$ we have:

$$(a_t\ \chi_{_M})b_s(r) = \begin{cases} \bigvee\{\ a_t\chi_{_M}(p_1) \wedge ... \wedge a_t\chi_{_M}(p_n) \wedge\ b_s(q_1) \wedge ... \wedge\ b_s(q_n)\}\ \ if\ \ r \in \sum_{i=1}^{n} p_i\gamma_i q_i, \\ 0 \qquad if \qquad otherwise, \end{cases}$$

(iv) This is proved Similar to (iii).

Theorem 4.9 *Let a_t, b_s are two arbitrary fuzzy points of R, for $a, b \in R,\ s, t \in [0, 1]$. Then*

(i) $\chi_{_M}\ \langle a_t\rangle\ \chi_{_M} = \chi_{_M}\ a_t\ \chi_{_M}$

(ii) $a_t\ \ \chi_{_M}\ b_s \subseteq \langle a_t\rangle\, \langle b_s\rangle$

Proposition 4.10 *For any fuzzy point a_t of R, we have $\chi_{_M} a_t \chi_{_M}$ belongs to $\eta-$ fuzzy prime hyperideal of R.*

Proof. using theorem 4.9 and the notion of product of two hyperideals, the proof is obvious.

Theorem 4.11 *Let $\mu, \nu \in FS(R)$ then*

(i) If $\mu \in \eta - FHI_r(R)$, then $\mu\nu \subseteq \mu$

(ii) If $\nu \in FHI_l(R)$, then $\mu\nu \subseteq \nu$.

Proof.

(i) Let $x \in R$

$$\begin{aligned}\mu\nu(x) &= \bigvee\left\{\mu(a_1)\wedge ... \wedge\mu(a_n)\wedge\nu(b_1)\wedge ... \wedge\nu(b_n) \mid x \in \sum_{i=1}^{n} a_i\gamma_i b_i, \quad a_i, b_i \in R\right\}\\ &\le \bigvee\left\{\mu(z_1)\wedge ... \wedge\mu(z_n)) \mid x \in \sum_{i=1}^{n} z_i \wedge \ z_i \in a_i\gamma_i b_i, \quad a_i, b_i \in R\right\}\\ &\le \bigvee\left\{\mu(t_i) \mid t_i \in \sum_{i=1}^{n} z_{ij}, \ z_{ij} \in a_j\gamma_j b_j, \ a_j, b_j \in R\right\}\\ &= \mu(x)\end{aligned}$$

Therefore $\mu\nu \subseteq \mu$.

(ii) This can be proved similar to (i).

Open Problem Is residual division fuzzy hyper ideal of hyperring, an Γ-hyperring? What conditions are required?

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