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On b-Chromatic Number of Various Middle and Total Graphs

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Abstract

A proper k-colorization of a graph $G = (V(G), E(G))$ is a mapping $l : V(G) \to N$ permits two neighboring vertices of distinct colors. The minimal of G has a proper k-coloring known as the $\chi(G)$-color number. A graph's b-colorization G is a type of proper k-colorization where a vertex is contained in each color class, where the vertices are of same color known as a vertex whose color dominates. The greatest integer k at which G allows b-coloring with k colors is known as the b-chromatic number. The b-chromatic number of the Middle and Total graph of Double wheel graph, Double Crown graph, Djembe graph, Sunflower graph, Closed Sunflower graph, Blossom graph and Butterfly graph with t number of vertices were all looked at in this study.

Keywords: *b-coloring; Double Wheel; Double Crown; Djembe; Sunflower; Closed Sunflower; Blossom; Butterfly graph.*

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1 Introduction

Colors C should be applied to the graph's vertices so as to distinguish them from each other. If $C = k$, C is described as a k-coloring. To color properly, we separate the vertices V into separate sets $\{V_1, V_2, V_3, .., V_k\}$. A color class is a set named V_r. The smallest integer $\chi(G)$ that enables G to properly color with $\chi(G)$ colors is known as G's chromatic number. When $\chi(G) = k$, the graph $\chi(G)$ is k-chromatic with k colors has a suitable b-coloring color classes which have neighbors in other color classes $(k-1)$. Color i is regulated by a colored vertex that has all its neighboring colors. The proper coloring for V is the one that is minimum in relation to a partial ordering specified on all of V's partitions. The b-chromatic number was proposed by Irving and Manlove in [9]. The maximum k for which a graph $\varphi(G)$ has a b-coloring by k colors is the graph's b-chromatic number.

We conclude that the chromatic number $\chi(G)$ is a lower bound for $\varphi(G)$ since every b-coloring is a proper coloring. For the upper bound, take note that each color class needs a b-vertex with a maximum of $\Delta(G)$ distinct colors nearby. The only additional color which is possible, is the color of a b-vertex itself. The smallest clique partition number and clique number of a graph provide an upper bound. Additionally, each bound's degree of tightness is noted. Inconsequentially upper bound for $\varphi(G)$ is $\Delta(G) + 1$. The resulting boundaries are as follows:
$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$.

The b-chromatic number of the cartesian product of two graphs was quantified by Kouider and Mah'eo [11], who also established lower and upper bounds. Kratochvil et al.[14] established that for connected bipartite graphs, the lower bound on the b-chromatic number is NP-complete with $k = \Delta(G)+1$. Additional proof that chordal graphs are b-continuous was given by Faik [8], who also emphasised the consistency of the b-coloring. According to Corteel et al. [3], there is no longer an inner approximation for the $120/133 - \epsilon$, b-chromatic broad range problem for any $\epsilon > 0$ until $P = NP$.

In respect to other graph properties, Kouider and Zaker [12] provided certain upper bounds for the b-chromatic number of particular types of graphs (clique number, chromatic number, biclique number). In [13], Kouider and El Sahili demonstrated that $\chi(G) = d + 1$ if G is a d-regular graph with girth 5 and no cycles of length 6. Discussions on the connections between this parameter and two other coloring parameters were suggested by Effantin and Kheddouci [7]. The dominating vertices in a b-coloring has a highly exciting quality because they can directly communicate with one another. Effantin et al. [6] proposed a distributed technique that employs a partitioning mechanism for a graph based on b-coloring to establish a particular coloring of graphs.

2 Preliminaries

We first review some basic concepts of graph theory.

The *Total graph* TG [1] of G, whose two vertices u, v in the vertex set of $T(G)$ are adjacent in $T(G)$ if one of the following holds:
(i) u, v are in $V(G)$ and u is adjacent to v in G.
(ii) u, v are in $E(G)$ and u, v are adjacent in G.
(iii) u is in $V(G)$, v is in $E(G)$, and u, v are incident in G.

The *Middle graph* MG [2] whose vertex set is $V(G) \cup E(G)$ where two vertices u, v are adjacent if,
(i) u, v are in $E(G)$ and u, v is in G.
(ii) One is a vertex of G and the other is an edge incident with it.
(iii) u is in $V(G)$, v is in $E(G)$ and u, v are incident in G.

The two disjoint cycles' vertices are joined to an external vertex to create the *Double Wheel graph* [15] $DW_t = 2C_t + k_1$. The *Double Crown graph* [10] C_t^{++}, is the graph obtained from the cycle C_t by attaching two pendent edges at each vertex of G. The *Djembe graph* [16] Dj_t is the graph created by connecting the corresponding vertices of two cycles of the same order t and connecting vertices of the two cycles to an outside vertex, $Dj_t = (C_t \square d_2) + k_1$. A *Sunflower graph* [4] SF_t is a graph generated by replacing each edge of a wheel graph W_t's rim with a triangle, providing that two triangles only share a vertex if and only if its corresponding vertices are adjacent in the original graph.

A *closed sunflower graph* [16] CSF_t is formed by merging the independent sunflower graph SF_t vertex points that are not adjacent to the central vertex and cause a cycle to occur on t vertex points. A closed sunflower graph CSF_t's outer cycle's vertices are connected to the centre vertex to create a *Blossom graph* [16] Bl_t. A planar undirected graph with 5 vertices and 6 vertices, the *Butterfly graph* [5] is denoted by the symbol BF. The hourglass graph and the Bowtie graph are other names for it. By connecting two similar cycle graphs C_3 at a common vertex, a butterfly graph can be created.

3 Main results

Theorem 3.1 *The b-chromatic number of TG of Double Wheel graph,* DW_t *and* $t \geq 3$ *is,*

$$\varphi\left(T(DW_t)\right) = (2t + 1).$$

Proof
Let $V(DW_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, TG on DW_t, each vertices of $a_r, b_r, a_0a_r, a_0b_r (1 \leq r \leq t)$ are segmented by the vertices $d_r, f_r, e_r, g_r (1 \leq r \leq t)$, respectively. Then,

$V[T(DW_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\}$
$\cup \{f_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\}$
Characterize $l : V[T(DW_t)] \to Z^+ - \{0\}$.
Mark the following $(2t+1)$ - colorings for $T(DW_t)$:
The apex vertex $l(a_0) = (2t+1)$,
For $1 \leq r \leq t$,
$l(a_r) = l(b_r) = l(g_r) = t + r$,
$l(e_r) = l(f_r) = r$,

$$l(d_r) = \begin{cases} r+2 & ; \quad 1 \leq r \leq (t-2) \\ 1 & ; \quad r = (t-1) \\ 2 & ; \quad r = t \end{cases}$$

Accordingly, $\varphi(T(DW_t)) \geq (2t+1)$. Considering that, $\varphi(T(DW_t))$ is more than $(2t+1)$, i.e., $\varphi(T(DW_t)) = (2t+2)$, then there must be at least $(2t+2)$ vertices of degree $(2t+1)$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices e_r, g_r, a_0 are having degree atleast $(2t+1)$. It's ludicrous, because b-coloring with $(2t+2)$ colors isn't attainable. As a result, $\varphi(T(DW_t)) \leq (2t+1)$. Hence, for $t \geq 3$, $\varphi(T(DW_t)) = (2t+1)$.

Theorem 3.2 *The b-chromatic number of MG of Double Wheel graph, $M(DW_t)$ and $t \geq 3$ is,*

$$\varphi(M(DW_t)) = (2t+1).$$

Proof
Let $V(DW_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, MG on DW_t, each edge of $a_r, b_r, a_0a_r, a_0b_r (1 \leq r \leq t)$ are segmented by the vertices $d_r, e_r, f_r, g_r (1 \leq r \leq t)$, respectively. Then,
$V[M(DW_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\}$
$\cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\}$
Characterize $l : V[M(DW_t)] \to Z^+ - \{0\}$.
Allocate the following $(2t+1)$ - colorings for $M(DW_t)$:
For $1 \leq r \leq t$ and $t \geq 3$,
The apex vertex $l(a_0) = (2t+1)$,

$$l(a_r) = \begin{cases} r+1 & ; \quad r < t \\ 1 & ; \quad r = t \end{cases} \qquad l(b_r) = \begin{cases} t+r+1 & ; \quad r < t \\ r+1 & ; \quad r = t \end{cases}$$

For $1 \leq r \leq t$,
$l(d_r) = l(g_r) = t + r, l(e_r) = l(f_r) = r$.

Accordingly, $\varphi(M(DW_t)) \geq (2t+1)$. Considering that, $\varphi(M(DW_t))$ is more than $(2t+1)$, i.e., $\varphi(M(DW_t)) = (2t+2)$, then there must be at least $(2t+2)$ vertices of degree $(2t+1)$ in $\varphi(M(DW_t))$, varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices b_t, q_t are having degree atleast $(2t+1)$.

It's ludicrous, because b-coloring with $(2t+2)$ colors isn't attainable. As a result, $\varphi\left(M(DW_t)\right) \leq (2t+1)$. Hence, for $t \geq 3$, $\varphi\left(M(DW_t)\right) = (2t+1)$.

Theorem 3.3 *The $b-$ chromatic number of MG of Double Crown graph, $M(C_t^{++})$ and $t \geq 7$ is,*

$$\varphi\left(M(C_t^{++})\right) = 7.$$

Proof

Let $V(C_t^{++}) = \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{c_r : 1 \leq r \leq t\}$.
Axiomatically, MG on C_t^{++}, each edge of $a_r : 1 \leq r \leq t$, $b_r : 1 \leq r \leq t$, $c_r : 1 \leq r \leq t$ are segmented by the vertices $d_r, e_r, f_r (1 \leq r \leq t)$, respectively.
Then,
$V[M(C_t^{++})] = \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{c_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\}$.
Characterize $l : V[M(C_t^{++})] \to \{1, 2, 3, 4, 5, 6, 7\}$.
Allocate the following t - colorings for $M(C_t^{++})$:
For $5 \leq t \leq 7, l(d_r) = r, 1 \leq r \leq t$,
For $t \geq 8$,

$$l(d_r) = \begin{cases} r & ; \quad \text{for } 1 \leq r \leq 7 \\ k & ; \quad \text{for } r \equiv k(mod 7), r \geq 8 \\ 3 & ; \quad r = t \end{cases}$$

and

$$m \equiv \begin{cases} 2 \pmod 5; l(d_{n-1}) = 1 \\ 3 \pmod 5; l(d_{n-1}) = 2; l(d_{n-2}) = 1 \\ 4 \pmod 5; l(d_{n-1}) = 4; l(d_{n-2}) = 2; l(d_{n-3}) = 1 \\ 5 \pmod 5; l(d_{n-1}) = 4; l(d_{n-2}) = 3; l(d_{n-3}) = 2; l(d_{n-4}) = 1 \\ 6 \pmod 5; l(d_{n-1}) = 5; l(d_{n-2}) = 4; l(d_{n-3}) = 3; l(d_{n-4}) = 2; l(d_{n-5}) = 1 \\ 0 \pmod 5; l(d_{n-1}) = 6; l(d_{n-2}) = 5; l(d_{n-3}) = 4; l(d_{n-4}) = 3; l(d_{n-5}) = 2; l(d_{n-6}) = 1 \end{cases}$$

For $5 \leq t \leq 7$,

$$l(a_r) = \begin{cases} r+2 & ; \quad \text{for } 1 \leq r \leq (t-2) \\ 1 & ; \quad \text{for } r = t-1 \\ 2 & ; \quad r = t \end{cases}$$

For $t \geq 8$,

$$l(a_r) = \begin{cases} r+6 & ; \quad \text{for } r = 1 \\ k+2 & ; \quad \text{for } r \equiv k(\bmod 7), r = 2, 3, 4, 5 \\ k-5 & ; \quad \text{for } r \equiv k(\bmod 7), r = 6, 7, 8 \\ 1 & ; \quad \text{for } r = t, r = 8, 15, 22, ... \end{cases}$$

For $5 \leq t \leq 6$,

$$l(e_r) = \begin{cases} r+1 & ; \quad \text{for } 1 \leq r \leq (t-1) \\ 1 & ; \quad r = t \end{cases}$$

For $t \geq 7$,

$$l(e_r) = \begin{cases} k+5 & ; \quad \text{for } r \equiv k(\bmod 7), r = 1,2 \\ k-2 & ; \quad \text{for } r \equiv k(\bmod 7), r = 3,4,5,6,7 \end{cases}$$

For $5 \leq t \leq 6$,
$l(f_1, f_2, f_3, f_4, f_5) = 4, 5, 1, 2, 3,$
$l(f_1, f_2, f_3, f_4, f_5, f_6) = 4, 5, 6, 1, 2, 3,$
For $t \geq 7$,

$$l(f_r) = \begin{cases} k+3 & ; \quad \text{for } r \equiv k(\bmod 7), r = 1,2,3,4 \\ k-4 & ; \quad \text{for } r \equiv k(\bmod 7), r = 5,6,7 \end{cases}$$

Accordingly, $\varphi\left(M(C_t^{++})\right) \geq 7$. Considering that, $\varphi\left(M(C_t^{++})\right)$ is more than 7, i.e., $\varphi\left(M(C_t^{++})\right) = 8$, then there must be at least 8 vertices of degree 7 in $\varphi\left(M(C_t^{++})\right)$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices d_r are having degree atleast 7. It's ludicrous, because b-coloring with 8 colors isn't attainable. As a result, $\varphi\left(M(C_t^{++})\right) \leq 7$. Hence, for $t \geq 3$, $\varphi\left(M(C_t^{++})\right) = 7$.

Theorem 3.4 *The b− chromatic number of TG of Djembe graph $T(Dj_t)$ and $t \geq 3$ is,*

$$\varphi\left(T(Dj_t)\right) = (2t+1).$$

Proof
Let $V(Dj_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, TG on Dj_t, each vetices of $a_r, b_r, a_rb_r, a_0a_r, a_0b_r (1 \leq r \leq t)$ are segmented by the vertices $d_r, g_r, e_r, f_r, h_r (1 \leq r \leq t)$, respectively. Then,
$V[T(Dj_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\} \cup \{h_r : 1 \leq r \leq t\}$
Characterize $l : V[T(Dj_t)] \to Z^+ - \{0\}$.
Mark the following $(2t+1)$ - colorings for $T(Dj_t)$:
The apex vertex $l(a_0) = (2t+1)$,
For $1 \leq r \leq t$,
$l(d_r) = l(h_r) = t + r$,
$l(b_r) = l(f_r) = r$,

$$l(a_r) = \begin{cases} t & ; \quad r = 1 \\ r-1 & ; \quad 2 \leq r \leq t \end{cases} \qquad l(g_r) = \begin{cases} r+2 & ; \quad 1 \leq r \leq (t-2) \\ 1 & ; \quad r = (t-1) \\ 2 & ; \quad r = t \end{cases}$$

$$l(e_r) = \begin{cases} t+r+1 & ; \quad 1 \leq r \leq (t-1) \\ t+1 & ; \quad r = t \end{cases}$$

Accordingly, $\varphi(T(Dj_t)) \geq (2t+1)$. Considering that, $\varphi(T(Dj_t))$ is more than $(2t+1)$, i.e., $\varphi(T(Dj_t)) = (2t+2)$, then there must be at least $(2t+2)$ vertices of degree $(2t+1)$ in $\varphi(T(Dj_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices f_r, h_r, a_0 are having degree atleast $(2t+1)$. It's ludicrous, because b-coloring with $(2t+2)$ colors isn't attainable. As a result, $\varphi(T(Dj_t)) \leq (2t+1)$. Hence, for $t \geq 3$, $\varphi(T(Dj_t)) = (2t+1)$.

Theorem 3.5 *The b− chromatic number of MG of Djembe graph $M(Dj_t)$ and $t \geq 3$ is,*

$$\varphi(M(Dj_t)) = (2t+1).$$

Proof
Let $V(Dj_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, MG on Dj_t, each edge of $a_r, b_r, a_0a_r, a_0b_r, a_rb_r (1 \leq r \leq t)$ are segmented by the vertices $d_r, e_r, f_r, g_r, h_r (1 \leq r \leq t)$, respectively. Then,
$V[M(Dj_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\} \cup \{h_r : 1 \leq r \leq t\}$
Characterize $l : V[M(Dj_t)] \to Z^+ - \{0\}$.
Allocate the following $(2t+1)$ - colorings for $M(Dj_t)$:
The apex vertex $l(a_0) = (2t+1)$,
For $1 \leq r \leq t$,
$l(a_r) = l(g_r) = t + r, l(e_r) = l(f_r) = r.$

$$l(b_r) = \begin{cases} r+1 & ; \quad r < t \\ 1 & ; \quad r = t \end{cases} \qquad l(h_r) = \begin{cases} t+r+1 & ; \quad r < t \\ r+1 & ; \quad r = t \end{cases}$$

$$l(d_r) = \begin{cases} r+2 & ; \quad 1 \leq r \leq (t-2) \\ 1 & ; \quad r = t-1 \\ 2 & ; \quad r = t \end{cases}$$

Accordingly, $\varphi(M(Dj_t)) \geq (2t+1)$. Considering that, $\varphi(M(Dj_t))$ is more than $(2t+1)$, i.e., $\varphi(M(Dj_t)) = (2t+2)$, then there must be at least $(2t+2)$ vertices of degree $(2t+1)$ in $\varphi(M(Dj_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices f_r, g_r are having degree atleast $(2t+1)$. It's ludicrous, because b-coloring with $(2t+2)$ colors isn't attainable. As a result, $\varphi(M(Dj_t)) \leq (2t+1)$. Hence, for $t \geq 3$, $\varphi(M(Dj_t)) = (2t+1)$.

Theorem 3.6 *The b- chromatic number of TG of Sunflower graph $T(SF_t)$ and $t \geq 3$ is,*

$$\varphi(T(SF_t)) = 2t.$$

Proof
Let $V(SF_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, TG on SF_t, each vertices of a_r, one side of a_r, another side of a_rb_r, $a_0a_r (1 \leq r \leq t)$ are segmented by the vertices $d_r, f_r, g_r, e_r (1 \leq r \leq t)$,

respectively. Then,
$V[T(SF_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\}$
Characterize $l : V[T(SF_t)] \to Z^+ - \{0\}$.
Mark the following $2t$ - colorings for $T(SF_t)$:
The apex vertex $l(a_0) = 1$,
For $1 \leq r \leq t$,
$l(b_r) = l(d_r) = r$,
$l(e_r) = t + r$,

$$l(a_r) = \begin{cases} r+2 & ; \quad 1 \leq r \leq (t-2) \\ 1 & ; \quad r = (t-1) \\ 2 & ; \quad r = t \end{cases} \qquad l(f_r) = \begin{cases} r+1 & ; \quad 1 \leq r \leq (t-1) \\ 1 & ; \quad r = t \end{cases}$$

$$l(e_r) = \begin{cases} t+r+1 & ; \quad 1 \leq r \leq (t-1) \\ t+1 & ; \quad r = t \end{cases}$$

$$l(g_r) = \begin{cases} 2t & ; \quad r = 1 \\ t+r-1 & ; \quad 2 \leq r \leq t \end{cases}$$

Accordingly, $\varphi(T(SF_t)) \geq 2t$. Considering that, $\varphi(T(SF_t))$ is more than $2t$, i.e., $\varphi(T(SF_t)) = (2t+1)$, then there must be at least $(2t+1)$ vertices of degree $2t$ in $\varphi(T(SF_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only vertices a_r, d_r are having degree atleast $2t$. It's ludicrous, because b-coloring with $(2t+1)$ colors isn't attainable. As a result, $\varphi(T(SF_t)) \leq 2t$. Hence, for $t \geq 3$, $\varphi(T(SF_t)) = 2t$.

Theorem 3.7 *The b- chromatic number of MG of Sunflower graph $M(SF_t)$ and $t \geq 3$ is,*

$$\varphi(M(SF_t)) = 2t$$

Proof
Let $V(SF_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, MG on SF_t, each edge of first side of b_r and second side of b_r, $a_r, a_0a_r(1 \leq r \leq t)$ are segmented by the vertices $d_r, e_r, f_r, g_r(1 \leq r \leq t)$, respectively. Then,
$V[M(SF_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\}$
Characterize $l : V[M(SF_t)] \to Z^+ - \{0\}$.
Allocate the following $2t$ - colorings for $M(SF_t)$:
The apex vertex $l(a_0) = 1$,
For $1 \leq r \leq t$,
$l(b_r) = l(f_r) = r, l(g_r) = t + r$,

$$l(a_r) = \begin{cases} t-r & ; \quad r = 1 \\ t & ; \quad r = 2 \\ j-2 & ; \quad 3 \leq r \leq t \end{cases} \qquad l(d_r) = \begin{cases} 2t & ; \quad r = 1 \\ t+r-1 & ; \quad 2 \leq r \leq t \end{cases}$$

$$l(e_r) = \begin{cases} 2t - r & ; \quad r = 1 \\ 2t & ; \quad r = 2 \\ t + r - 2 & ; \quad 3 \leq r \leq t \end{cases}$$

Accordingly, $\varphi(M(SF_t)) \geq 2t$. Considering that, $\varphi(M(SF_t))$ is more than $2t$, i.e., $\varphi(M(SF_t)) = (2t+1)$, then there must be atleast $(2t+1)$ vertices of degree $2t$ in $\varphi(M(SF_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices f_r are having degree atleast $(2t+1)$. It's ludicrous, because b-coloring with $2t$ colors isn't attainable. As a result, $\varphi(M(SF_t)) \leq 2t$. Hence, for $t \geq 3$, $\varphi(M(SF_t)) = 2t$.

Theorem 3.8 *The $b-$ chromatic number of TG of Closed Sunflower graph $T(CSF_t)$ and $t \geq 3$ is,*

$$\varphi(T(CSF_t)) = 2t.$$

Proof

Let $V(CSF_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.

Axiomatically, TG on CSF_t, each vertices of a_r, one side of a_rb_r, another side of a_rb_r, $a_0a_r, b_r(1 \leq r \leq t)$ are segmented by the vertices $d_r, f_r, g_r, e_r, h_r(1 \leq r \leq t)$, respectively. Then,

$V[T(CSF_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\} \cup \{h_r : 1 \leq r \leq t\}$

Characterize $l : V[T(CSF_t)] \to Z^+ - \{0\}$.

Mark the following $2t$ - colorings for $T(CSF_t)$:

The apex vertex $l(a_0) = 2t$,

For $1 \leq r \leq t$,

$l(a_r) = l(h_r) = r$,

$l(d_r) = t + r$,

$$l(b_r) = \begin{cases} 2t & ; \quad r = 1 \\ t + r - 1 & ; \quad 2 \leq r \leq t \\ 2 & ; \quad r = t \end{cases} \qquad l(e_r) = \begin{cases} r + 2 & ; \quad 1 \leq r \leq (t-2) \\ 1 & ; \quad r = (t-1) \\ 2 & ; \quad r = t \end{cases}$$

$$l(g_r) = \begin{cases} t + r + 2 & ; \quad 1 \leq r \leq (t-2) \\ t + 1 & ; \quad r = (t-1) \\ t + 2 & ; \quad r = t \end{cases}$$

For $1 \leq r \leq t$,

$l(f_r) = r, r = t, t-1, t-2, ..., 1$.

Accordingly, $\varphi(T(CSF_t)) \geq 2t$. Considering that, $\varphi(T(CSF_t))$ is more than $2t$, i.e., $\varphi(T(CSF_t)) = (2t+1)$, then there must be at least $(2t+1)$ vertices of degree $2t$ in $\varphi(G(CSF_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only vertices a_r, d_r are having degree $2t$. It's ludicrous, because b-coloring with $(2t+1)$ colors isn't attainable. As a result, $\varphi(T(CSF_t)) \leq 2t$. Hence, for $t \geq 3$, $\varphi(T(CSF_t)) = 2t$.

Theorem 3.9 *The b− chromatic number of MG of Closed Sunflower graph $M(CSF_t)$ and $t \geq 3$ is,*

$$\varphi\left(M(CSF_t)\right) = 2t.$$

Proof

Let $V(CSF_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, MG on CSF_t, each edge of first side of b_r and second side of b_r, $a_r, b_r, a_0a_r(1 \leq r \leq t)$ are segmented by the vertices $d_r, e_r, f_r, g_r, h_r(1 \leq r \leq t)$, respectively. Then,
$V[M(CSF_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\} \cup \{h_r : 1 \leq r \leq t\}$
Characterize $l : V[M(CSF_t)] \to Z^+ - \{0\}$.
Allocate the following $2t$ - colorings for $M(CSF_t)$:
The apex vertex $l(a_0) = 1$,
For $1 \leq r \leq t$,
$l(b_r) = l(f_r) = r, l(g_r) = l(h_r) = t + r$,

$$l(a_r) = \begin{cases} t - r & ; \quad r = 1 \\ t & ; \quad r = 2 \\ r - 2 & ; \quad 3 \leq r \leq t \end{cases} \qquad l(d_r) = \begin{cases} 2t & ; \quad r = 1 \\ t + r - 1 & ; \quad 2 \leq r \leq t \end{cases}$$

$$l(e_r) = \begin{cases} 2t - r & ; \quad r = 1 \\ 2t & ; \quad r = 2 \\ t + r - 2 & ; \quad 3 \leq r \leq t \end{cases}$$

Accordingly, $\varphi\left(M(CSF_t)\right)) \geq 2t$. Considering that, $\varphi\left(M(CSF_t)\right)$ is more than $2t$, i.e., $\varphi\left(M(CSF_t)\right) = (2t + 1)$, then there must be atleast $(2t + 1)$ vertices of degree $2t$ in $\varphi\left(M(CSF_t)\right)$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices d_r are having degree atleast $2t$. It's ludicrous, because b-coloring with $(2t + 1)$ colors isn't attainable. As a result, $\varphi\left(M(CSF_t)\right) \leq 2t$. Hence, for $t \geq 3$, $\varphi\left(M(CSF_t)\right) = 2t$.

Theorem 3.10 *The b− chromatic number of TG of Blossom graph $T(Bl_t)$ and $t \geq 3$ is,*

$$\varphi\left(T(Bl_t)\right) = 2t.$$

Proof

Let $V(Bl_t) = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\}$.
Axiomatically, TG on Bl_t, each vertices of a_r, one side of a_rb_r, another side of a_rb_r, $a_0a_r(1 \leq r \leq t)$ are segmented by the vertices $d_r, f_r, g_r, e_r, (1 \leq r \leq t)$, respectively. Then,
$V[T(Bl_t)] = \{a_0\} \cup \{a_r : 1 \leq r \leq t\} \cup \{b_r : 1 \leq r \leq t\} \cup \{d_r : 1 \leq r \leq t\} \cup \{e_r : 1 \leq r \leq t\} \cup \{f_r : 1 \leq r \leq t\} \cup \{g_r : 1 \leq r \leq t\}$
Characterize $l : V[T(Bl_t)] \to Z^+ - \{0\}$.
Mark the following $2t$ - colorings for $T(Bl_t)$:
The apex vertex $l(a_0) = 1$,

For $1 \leq r \leq t$,
$l(b_r) = l(d_r) = r$,
$l(f_r) = t + r$,

$$l(a_r) = \begin{cases} t+r+1 & ; \; 1 \leq r \leq (t-1) \\ t+1 & ; \; r = t \end{cases} \qquad l(e_r) = \begin{cases} 2t & ; \; r = 1 \\ t+r-1 & ; \; 2 \leq r \leq t \end{cases}$$

For $1 \leq r \leq t$,

$$l(g_r) = \begin{cases} t, t-1, ..., t-(r-1) & \\ 1 & ; \; r = (t-1) \\ 2 & ; \; r = t \end{cases}$$

Accordingly, $\varphi\,(T(Bl_t)) \geq 2t$. Considering that, $\varphi\,(T(Bl_t))$ is more than $2t$, i.e., $\varphi\,(T(Bl_t)) = (2t+1)$, then there must be at least $(2t+1)$ vertices of degree $2t$ in $\varphi\,(G(Bl_t))$, each with a varied color adjacent to the vertices of all other colors for $t \geq 3$. But, only the vertices a_r, d_r are having degree atleast $2t$. It's ludicrous, because b-coloring with $(2t+1)$ colors isn't attainable. As a result, $\varphi\,(T(Bl_t)) \leq 2t$. Hence, for $t \geq 3$, $\varphi\,(T(Bl_t)) = 2t$.

Theorem 3.11 *The $b-$ chromatic number of TG of Butterfly graph $T(BF_3)$ is,*

$$\varphi\,(T(BF_3)) = 5.$$

Proof
Let $V(BF_3) = \{a_0\} \cup \{a_1, a_2\} \cup \{b_1, b_2\}$.
Axiomatically, TG on BF_3, vertices of $a_1a_2, a_0a_1, a_0a_2, b_1b_2, a_0b_1, a_0b_2$ are segmented by the vertices $d_1, d_2, d_3, e_1, e_2, e_3$, respectively. Then,
$V[T(BF_3)] = \{a_0\} \cup \{a_1, a_2\} \cup \{b_1, b_2\} \cup \{d_1, d_2, d_3\} \cup \{e_1, e_2, e_3\}$
Characterize $l : V[T(BF_3)] \rightarrow \{1, 2, 3, 4, 5\}$.
Mark the following 5 - colorings for $T(BF_3)$:
The apex vertex $l(a_0) = 5$,
$l(a_1) = 2, l(a_2) = 1$,
$l(b_1) = 4, l(b_2) = 3$,
$l(d_1) = 5, l(d_2) = 1, l(d_3) = 2$,
$l(e_1) = 6, l(e_2) = 3, l(e_3) = 4$,

Accordingly, $\varphi\,(T(BF_3)) \geq 5$. Considering that, $\varphi\,(T(BF_3))$ is more than 5, i.e., $\varphi\,(T(BF_3)) = 6$, then there must be at least 6 vertices of degree 5 in $\varphi\,(T(BF_3))$, each with a varied color adjacent to the vertices of all other colors. But, only the vertices d_2, d_3, a_0, e_2e_3 are having degree atleast 5. It's ludicrous, because b-coloring with 6 colors isn't attainable. As a result, $\varphi\,(T(BF_3)) \leq 5$. Hence, $\varphi\,(T(BF_3)) = 5$.

4 Open Problem

In this paper, for any natural number $t \geq 3$, the b-chromatic number of middle and total graph of Double Wheel, Double Crown, Djembe, Sunflower, Closed Sunflower, Blossom, Butterfly graph is equal to $2t$ or $2t+1$. However, the exact value of b-chromatic number of Middle and Total graph of any graph G is still Open Problem.

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