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The SGEM Method For Solving Some Time and Space-Conformable Fractional Evolution Problems

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Abstract: *We apply the sine-Gordon expansion method, SGEM for short, to derive solutions for some classes of time and space-conformable fractional evolution equations. Some examples are discussed and their graphs are plotted to illustrate the efficiency of the SGEM method.*

Keywords: *SGEM method, traveling wave solution, time and space equation, Khalil derivative, evolution equation.*

2010 Mathematics Subject Classification: 35C07, 35R15, 35Q40.

1 Introduction

Fractional evolution differential play an important role in many research fields such as fluid mechanics, fluid flow, and other areas. We refer the reader to the papers [1, 7, 9, 2, 5, 6, 8, 10, 12, 29, 30] for some other applications. Several methods have been used to study solutions of evolution equations, like for instance, VIM method [17, 18, 19], ADM method [3, 4], Tanh-Coth method [20, 21, 22], Taylor and finite difference methods [14, 15, 16], and Homotopy perturbation method [13, 31].
In this paper, we apply the SGEM method [11, 27] to solve some classes of conformable fractional evolution problems of the form:

$$AT_t^{2\alpha}u\left(x,t\right)+BT_x^{2\beta}u\left(x,t\right)=f\left(u,T_t^{\alpha}u,T_x^{\beta}u\right), \tag{1}$$

under the conditions that $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivatives in the sense of Khalil [26]; with $0<\alpha,\beta\leq 1$, $A,B\in\mathbb{R}$ and $f:\mathbb{R}^3\to\mathbb{R}$ is a

given function.
The use of Khalil derivatives in the evolution problems of the present paper can be motivated by the fact that these derivatives are important in differential equations, see for example [2, 28].

2 Conformable Fractional Derivatives

We refer the reader to [23, 24, 25, 26] for more information on this section.

Definition 1 *Let* $f:(0,\infty)\to\mathbb{R}$*. The conformable fractional derivative is written as*

$$(T^{\alpha}f)(x)=\frac{\partial^{\alpha}f(x)}{\partial x^{\alpha}}=\lim_{\varepsilon\to 0}\left(\frac{f\left(x+\varepsilon x^{1-\alpha}\right)-f(x)}{\varepsilon}\right),\ x>0,\alpha\in(0,1]. \tag{2}$$

We recall the following properties:
If f *and* g *are two functions,* $a,b\in\mathbb{R}$*,* $x>0$ *and* $\alpha\in(0,1]$*, then we have:*

- $$T^{\alpha}(af(x)+bg(x))=aT^{\alpha}(f(x))+bT^{\alpha}(g(x)).$$
- $$T^{\alpha}(f(x)g(x))=g(x)T^{\alpha}(f(x))+f(x)T^{\alpha}(g(x)).$$
- $$T^{\alpha}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)T^{\alpha}(f(x))-f(x)T^{\alpha}(g(x))}{(g(x))^{2}},\quad g\neq 0.$$
- $$(T^{\alpha}f)(x)=x^{1-\alpha}\frac{df}{dx}.$$
- $$T^{\alpha}\lambda=0,\qquad \textit{for all constant } \lambda.$$
- $$T^{\alpha}(x^{p})=kx^{p-\alpha},\qquad \textit{for all } p\in\mathbb{R}.$$

3 The SGEM Method

Now, we recall the SGEM method for a generalized class of Khalil conformable fractional derivatives. The following steps can be found in any paper dealing with evolution equations by means of SGEM method, see for instance [11, 27].

Let us consider "the Sine-Gordon equation" involving time and space-Khalil derivatives:

$$T_x^{2\beta}\left(u\left(x,t\right)\right)-T_t^{2\alpha}\left(u\left(x,t\right)\right)=m^2\sin\left(u\left(x,t\right)\right), \tag{3}$$

where $u\left(x,t\right)$ is a function and $m\neq 0$ is a real constant.
We use the wave transform given by

$$U\left(\xi\right)=u\left(t,x\right)\quad ,with\xi=k\left(\frac{x^\beta}{\beta}-c\frac{t^\alpha}{\alpha}\right). \tag{4}$$

It is to note in the paper, the authors have used a similar transform but without parameter β. Then, we reduce (3) to the following equation:

$$\frac{\partial^2 U}{\partial\xi^2}=\frac{m^2}{k^2(1-c^2)}\sin U, \tag{5}$$

where $k\in\mathbb{R}$ [30].
So, we have

$$\left(\frac{\partial\left(\frac{U}{2}\right)}{\partial\xi}\right)^2=\frac{m^2}{k^2(1-c^2)}\sin^2\left(\frac{U}{2}\right)+C, \tag{6}$$

where C is a constant of integration which can be taken equal to 0.
Let $w(\xi)=\frac{U(\xi)}{2}$ and $b^2=\frac{m^2}{k^2(1-c^2)}$, and by using (6), we obtain:

$$\frac{\partial(w)}{\partial\xi}=b\sin\left(w\right), \tag{7}$$

For $b=1$ in (7), we transform (7) to:

$$\sin w\left(\xi\right)=\left.\frac{2\gamma e^\xi}{\gamma^2e^{2\xi}+1}\right|_{\gamma=1}=\sec h\xi, \tag{8}$$

or

$$\cos w\left(\xi\right)=\left.\frac{\gamma^2e^{2\xi}-1}{\gamma^2e^{2\xi}+1}\right|_{\gamma=1}=\tanh\xi, \tag{9}$$

Any general problem of type:

$$P\left(u,T_t^\alpha u,T_x^\beta u,T_t^{2\alpha}u,T_x^{2\beta}u,...\right)=0 \tag{10}$$

is then transformed into the following general form equation:

$$\widetilde{P}\left(U,U',U",...\right)=0. \tag{11}$$

The solution of (11) is then presented by the expression:

$$U\left(\xi\right)=a_0+\sum_{i=1}^{n}\tanh^{i-1}\left(\xi\right)\left(a_i\tanh\left(\xi\right)+b_i\sec h\left(\xi\right)\right), \tag{12}$$

so that

$$U\left(w(\xi)\right)=a_0+\sum_{i=1}^{n}\cos^{i-1}\left(w(\xi)\right)\left(a_i\cos\left(w(\xi)\right)+b_i\sin h\left(w(\xi)\right)\right), \tag{13}$$

such that a_0, a_i, b_i $(1\leq i\leq n\ )$ and n is fixed.
To determine the constants, one needs to equate coefficients of $\sin^p\left(w(\xi)\right)\cos^q\left(w(\xi)\right)$ with zero to obtain an algebraic system.
Replacing these results in (12), we immediately obtain the solutions to the evolution problem (10).

4 Two Examples on Conformable Fractional Evolution Problems

Let's consider the conformable fractional evolution problem:

$$AT_t^{2\alpha}u\left(x,t\right)+BT_x^{2\beta}u\left(x,t\right)=f\left(u,T_t^{\alpha}u,T_x^{\beta}u\right).$$

Thanks to the change of variable

$$u\left(t,x\right)=U\left(\xi\right),\qquad \xi=k\left(\tfrac{x^\beta}{\beta}-c\tfrac{t^\alpha}{\alpha}\right), \tag{14}$$

the above evolution problem can be written as:

$$\left(Ak^2+Bk^2c^2\right)U_{\xi\xi}=g\left(U,U_\xi\right), \tag{15}$$

where $g:\mathbb{R}^2\rightarrow\mathbb{R}$ is a function with $g\left(U,U_\xi\right)=f\left(U,-kcU_\xi,kU_\xi\right)$.
By the SGEM method, the solutions of (15) are the following:

$$U\left(\xi\right)=a_0+\sum_{i=1}^{n}\tanh^{i-1}\left(\xi\right)\left(a_i\tanh\left(\xi\right)+b_i\sec h\left(\xi\right)\right), \tag{16}$$

where $a_i, b_i\in\mathbb{R}$.

4.1 Example 1

We begin by considering the time and space evolution equation:

$$T_t^{2\alpha}u\left(x,t\right)+pT_t^{\alpha}u\left(x,t\right)+quT_x^{\beta}u=0. \tag{17}$$

Thanks to (14), the equation in (17) becomes:

$$k^2c^2U_{\xi\xi}-kcpU_\xi+qkUU_\xi=0. \tag{18}$$

For this case, we have $n = 1$, and then the solution of (18) is

$$U(w) = a_0 + a_1 \cos(w) + b_1 \sin(w). \tag{19}$$

Therefore,

$$(k^2c^2b_1 + qka_1b_1)\cos^2 w \sin w + (qka_0b_1 - kcpb_1)\cos w \sin w$$

$$+ (qkb_1^2 - qka_1^2 - 2k^2c^2a_1)\cos w \sin^2 w$$

$$- (k^2c^2b_1 + qkb_1a_1)\sin^3 w + (kcpa_1 - qka_0a_1)\sin^2 w = 0$$

We find the system of algebraic type:

$$\begin{cases} k^2c^2b_1 + qka_1b_1 = 0 \\ qka_0b_1 - kcpb_1 = 0 \\ qkb_1^2 - qka_1^2 - 2k^2c^2a_1 = 0 \\ k^2c^2b_1 + qkb_1a_1 = 0 \\ kcpa_1 - qka_0a_1 = 0 \end{cases}$$

It can be solved using Maple. Thus, we obtain the following families of solutions:

Family 1

$$a_0 = a_0, a_1 = 0, b_1 = 0, k = k, c = c,$$

$$u(x,t) = a_0. \tag{20}$$

Family 2

$$a_0 = \tfrac{cp}{q}, a_1 = -\tfrac{2kc^2}{q}, b_1 = 0, k = k, c = c,$$

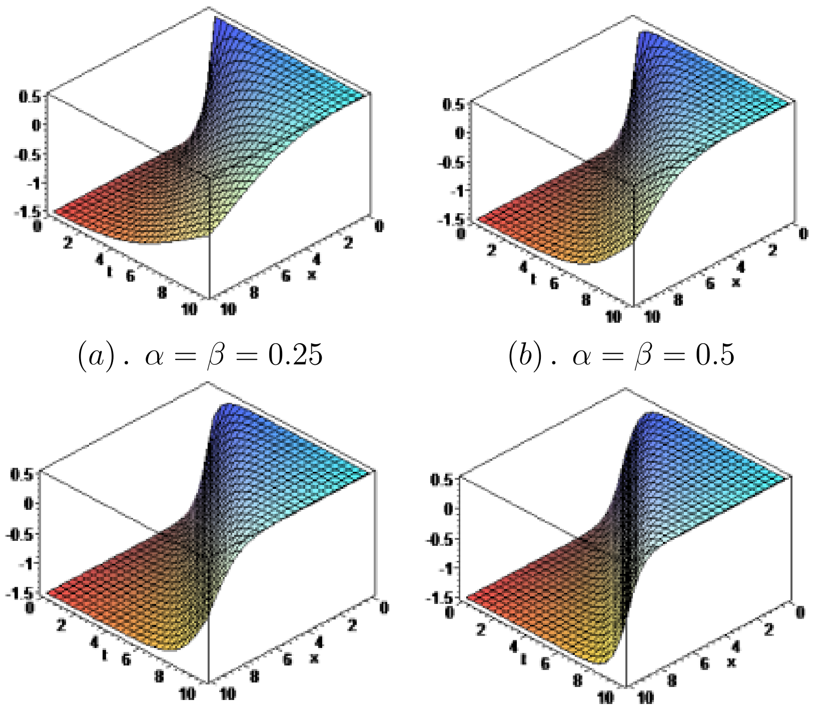
$$u(x,t) = \tfrac{cp}{q} - \tfrac{2kc^2}{q}\tanh\left(k\left(\tfrac{x^\beta}{\beta} - c\tfrac{t^\alpha}{\alpha}\right)\right) \tag{21}$$

Family 3

$$a_0 = \tfrac{cp}{q}, a_1 = a_1, b_1 = b_1, k = 0, c = c,$$

$$u(x,t) = \tfrac{cp}{q} + b_1 \tag{22}$$

Figure 4.1 indicates the graph of (21) of Eq.(17) under different values of $(\alpha,\beta) \in \{(0.25,0.25),(0.5,0.5),(0.75,0.75),(1,1)\}$, with $k=1, c=1, p=-1$ and $q=2$.



(a). $\alpha=\beta=0.25$ (b). $\alpha=\beta=0.5$

(c). $\alpha=\beta=0.75$ (d). $\alpha=\beta=1$

Figure 4.1: The graph representation of (21)

4.2 Example 2

The second considered example is the following:

$$T_x^{2\beta}u(x,t) + pT_x^{\beta}u(x,t) + quT_t^{\alpha}u = 0. \tag{23}$$

Using (14), so we can transform (23) into the following equation:

$$k^2U_{\xi\xi} + pkU_\xi - qckUU_\xi = 0. \tag{24}$$

By the balance technique on UU_ξ and $U_{\xi\xi}$, we have $n=1$. So, the solution of (24) has the form:

$$U(w) = a_0 + a_1\cos(w) + b_1\sin(w). \tag{25}$$

Hence,

$$kb_1(k+p-qca_0)\cos^2 w\sin w+2ka_1(qca_0-k-p)\cos w\sin^2 w$$
$$+kb_1(qca_0-k-p)\sin^3 w+qck\left(2a_1^2-b_1^2\right)\cos^2 w\sin^2 w$$
$$-qcka_1b_1\cos^3 w\sin w+3qcka_1b_1\cos w\sin^3 w+qckb_1^2\sin^4 w=0$$

Therefore, we obtain the system:

$$\begin{cases} kb_1(k+p-qca_0)=0\\ ka_1(qca_0-k-p)=0\\ qck\left(2a_1^2-b_1^2\right)=0\\ qcka_1b_1=0\\ qckb_1^2 \end{cases}$$

Thus, we obtain the following sets of solutions:
Family 1

$$a_0=a_0, a_1=0, b_1=0, k=k, c=c,$$

$$u(x,t)=a_0. \tag{26}$$

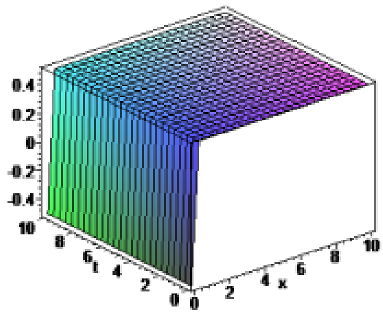
Family 2

$$a_0=a_0, a_1=a_1, b_1=b_1, , k=-p, c=0,$$

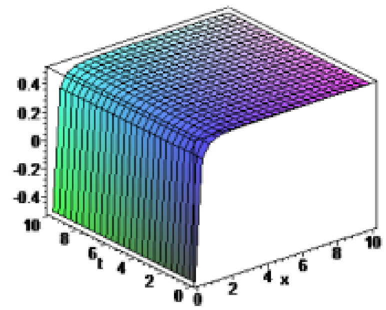
$$u(x,t)=a_0+a_1\tanh\left(-p\left(\frac{x^\beta}{\beta}\right)\right)+b_1\sec h\left(-p\left(\frac{x^\beta}{\beta}\right)\right). \tag{27}$$

Figure 4.2 indicates the graph of (27) for (23) under the values of $(\alpha,\beta)\in\{(0.25,0.25),(0.5,0.5),(0.75,0.75),(1,1)\}$, with $a_0=-\frac{1}{2}, a_1=-1, b_1=0, p=$

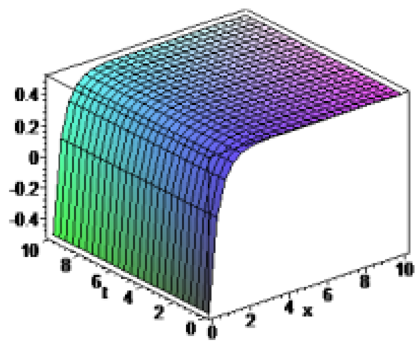
-1 and $q = 2$.



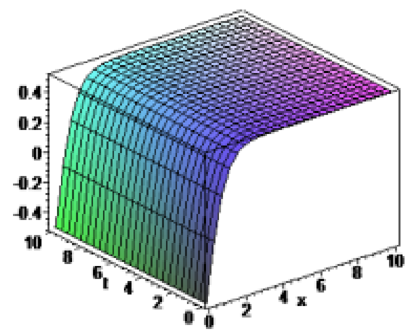
(a) $\alpha = \beta = 0.25$



(b) $\alpha = \beta = 0.5$



(c) $\alpha = \beta = 0.75$



(d) $\alpha = \beta = 1$

Figure 4.2: Graph representation of (27)

5 Conclusion and Open Problem

We have obtained new solutions for two classes of time and space-conformable fractional evolution equations by using the SGEM method. Maple has been used to solve the obtained algebraic systems. The obtained solutions and their graphs show that the SGEM method can be effectively used to solve nonlinear conformable fractional evolution problems.

At the end of this paper, we shall propose the following open question:

We think it is important to address a comparative study between the ADM, the VIM and the Tanh methods for the above general conformable fractional evolution problem.

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