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Some results on generalized derivations involving prime ideals

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Abstract

Let R ***be a ring,*** P ***a prime ideal of*** R ***and*** $F : R \to R$ ***a generalized derivation associated with a derivation*** d ***of*** R***. If any one of the following holds then*** $d(R) \subseteq P$ ***or*** R/P ***is commutative integral domain: i)*** $[F(x), x] \in P$, ***ii)*** $F([x, y]) \in P$, ***iii)*** $F(xoy) \in P$, ***iv)*** $F([x, y]) \pm [x, y] \in P$, ***v)*** $F(xoy) \pm (xoy) \in P$, ***vi)*** $F([x, y]) \pm (xoy) \in P$, ***vii)*** $F(xoy) \pm [x, y] \in P$, ***viii)*** $F([x, y]) \pm [F(x), y] \in P$, ***ix)*** $F(xoy) \pm (F(x)oy) \in P$, ***x)*** $F([x, y]) \pm x^m[x, y]x^n \in P$, ***xi)*** $F(xoy) \pm x^m(xoy)x^n \in P$, ***xii)*** $F([x, y]) \pm x^m(xoy)x^n \in P$, ***xiii)*** $F(xoy) \pm x^m[x, y]x^n \in P$, ***for all*** $x, y \in R, m, n \in \mathbb{Z}$.

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1 Introduction

Let R will be an associative ring with center Z. Recall that a proper ideal P of R is said to be prime if for any $x, y \in R, xRy \subseteq P$ implies that $x \in P$ or $y \in P$. The ring R is prime if and only if (0) is a prime ideal of R. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $xoy = xy + yx$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. This is the very first example of derivation. More generally, an additive mapping $F : R \to R$ is called a generalized derivation associated with d if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

For fixed $a, b \in R$, a typical example of a generalized derivation is the mapping $x \to ax + xb$, which is called the generalized inner derivation induced by a and b, with associated derivation $x \to [x, a]$. This definition was given by M. Bresar in [6]. One may observe that the concept of generalized derivations includes the concept of derivations and of the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). Generalized derivations have been primarily studied on operator algebras.

Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. The study of such mappings was initiated by E. C. Posner in [3]. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. In [7], Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R contains a nonzero central ideal. This theorem considered for generalized derivations by Quadri et al. in [8] and extended by Dhara proving $F([x, y]) \pm [x, y] \in Z$, for all $x, y \in I$, when F is a generalized derivation of R in [2].

Recently, some authors adopted a new approach by considering algebraic identties with derivations involving prime ideal without primenees assumption on the considered ring (see, e.g., [4], [1], [5], [9] and references therein). They characterized the commutativity of a quotient ring R/P. The derivations and their generalizations play major role in mathematics, economics, quantum physics and biology such as chemotheraphy. Beceuse of that the correlation between derivations and the algebraic structures of quotient rings has become an exciting subject in the the last years.

In the present paper is motivated by the previous results. We aim to investigate the commutativity of quotient ring R/P where R any ring and P is prime ideal of R which admits generalized derivations are satisfying some identities acting on prime ideals P.

2 Results

Throughout the paper, we will make some extensive use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$

$[xy, z] = [x, z]y + x[y, z]$
$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$
$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$

Lemma 2.1 *[9, Lemma 1.3]Let R be a ring, P a prime ideal of R. If any of the following conditions is satisfied for all $x, y \in R$, then R/P is commutative integral domain.*
i) $[x, y] \in P$
ii) $xoy \in P$

Theorem 2.2 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $[F(x), x] \in P$, for all $x \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By the hypothesis, we have

$$[F(x), x]) \in P, \quad \text{for all } x \in R. \tag{1}$$

Linearizing (1) gives

$$y[x, d(x)] + [x, y]d(x) \in P, \quad \text{for all } x, y \in R. \tag{2}$$

Writting $zy, z \in R$ for y in (2) and using (2), we obtain that

$$[x, z]yd(x) \in P, \quad \text{for all } x, y, z \in R.$$

Therefore, we have

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R. \tag{3}$$

Since P is prime, we get

$$[x, z] \in P \text{ or } d(x) \in P, \quad \text{for all } x, z \in R. \tag{4}$$

Let $L = \{x \in R \mid [x, z] \in P, \text{ for all } z \in R.\}$ and $K = \{x \in R \mid d(x) \in P\}$. Clearly each of L and K is additive subgroup of R such that $R = L \cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L = R$ or $K = R$. In the first case, we have $[x, z] \in P$, for all $z \in R$, and so R/P is an integral domain by Lemma 1. In the second case, we get $d(R) \subseteq P$. This completes the proof.

Theorem 2.3 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F([x, y]) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$F([x,y]) \in P, \quad \text{for all } x,y \in R. \tag{5}$$

Writting yx for y in (5) and using this, we obtain that

$$[x,y]d(x) \in P, \quad \text{for all } x,y \in R. \tag{6}$$

Taking $zy, z \in R$ for y in (6) and using (6), we get

$$[x,z]Rd(x) \subseteq P, \quad \text{for all } x,z \in R.$$

Using the same arguments after (3) in the proof of Theorem 1, we get the required result.

Theorem 2.4 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F(xoy) \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$F(xoy) \in P, \quad \text{for all } x,y \in R. \tag{7}$$

Writting yx for y in (7) and using this, we find that

$$(xoy)d(x) \in P, \ \text{for all } x,y \in R. \tag{8}$$

Substituting $yz, z \in R$ for y in (8) and using this expression, we arrive at

$$[x,z]Rd(x) \subseteq P, \quad \text{for all } x,z \in R.$$

Arguing the same methods after (3) in the proof of Theorem 1, we obtain the required result.

Theorem 2.5 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F([x,y]) \pm [x,y] \in P$, for all $x,y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $F = 0$, then we get

$$[x,y] \in P, \quad \text{for all } x,y \in R.$$

By Lemma 1, we get R/P is commutative integral domain.
Now, we assume that $F \neq 0$. By our hypothesis, we have

$$F([x,y]) \pm [x,y] \in P, \quad \text{for all } x,y \in R. \tag{9}$$

Replacing y by yx in (9) and using this equation, we arrive that

$$[x, y]d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $yz, z \in R$ for y in this expression and using this, we get

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Using the same arguments after (3) in the proof of Theorem 1, we obtain the required result.

Theorem 2.6 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F(xoy) \pm (xoy) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $F = 0$, then we get

$$xoy \in P, \quad \text{for all } x, y \in R.$$

By Lemma 1, we get R/P is commutative integral domain. Now, we get $F \neq 0$. By our hypothesis, we have

$$F(xoy) \pm (xoy) \in P, \quad \text{for all } x, y \in R. \tag{10}$$

Replacing y by yx in this equation and using this, we arrive that

$$(xoy)d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $yz, z \in R$ for y and using this equation, we find that

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

We obtain the the required result using the same arguments after (3) in the proof of Theorem 1.

Theorem 2.7 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F([x, y]) \pm (xoy) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. If $F = 0$, then we get

$$xoy \in P, \quad \text{for all } x, y \in R.$$

Hence R/P is commutative integral domain by Lemma 1.
Now, we assume $F \neq 0$. By our hypothesis, we have

$$F([x, y]) \pm (xoy) \in P, \quad \text{for all } x, y \in R. \tag{11}$$

Replacing yx by y in (11) and using this, we get

$$[x,y]d(x)\in P,\quad \text{for all } x,y\in R.$$

Taking $zy, z\in R$ for y in this equation and using this, we have

$$[x,z]Rd(x)\subseteq P,\quad \text{for all } x,z\in R.$$

This equation is the same as (3). Arguing the same lines in the proof of Theorem 1, we get the required result.

Theorem 2.8 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If$F(xoy)\pm[x,y]\in P$, for all $x,y\in R$, then $d(R)\subseteq P$ or R/P is commutative integral domain.*

Proof. If $F=0$, then $[x,y]\in P$, for all $x,y\in R$. By Lemma 1, we find that R/P is commutative integral domain.
Now, we have $F\neq 0$ and

$$F(xoy)\pm[x,y]\in P,\quad \text{for all } x,y\in R. \tag{12}$$

Replacing yx by y in (12) and using this, we get

$$(xoy)d(x)\in P,\quad \text{for all } x,y\in R.$$

Substituting $yz, z\in R$ for y in this equation and using this, we find that

$$[x,z]Rd(x)\subseteq P,\quad \text{for all } x,z\in R.$$

Using the same arguments after (3) in the proof of Theorem 1, we obtain the required result.

Theorem 2.9 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F([x,y])\pm[F(x),y]\in P$, for all $x,y\in R$, then $d(R)\subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$F([x,y])\pm[F(x),y]\in P,\quad \text{for all } x,y\in R. \tag{13}$$

Replacing y by yx in (13) and using this equation, we arrive that

$$[x,y]d(x)\pm y[F(x),x]\in P,\quad \text{for all } x,y,z\in R. \tag{14}$$

Writing $zy, z\in R$ instead of y in (14) and using (14), we have

$$[x,z]yd(x)\in P,\quad \text{for all } x,y,z\in R$$

Arguing the same methodts after (3) in the proof of Theorem 1, we get the required result.

Theorem 2.10 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F(xoy) \pm (F(x)oy) \in P$, for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By our hypothesis, we get

$$F(xoy) \pm (F(x)oy) \in P, \quad \text{for all } x, y \in R. \tag{15}$$

Writting yx for y in (15) and using this, we obtain that

$$(xoy)d(x) \mp y[F(x), x] \in P, \quad \text{for all } x, y \in R. \tag{16}$$

Substituting $zy, z \in R$ for y in (16) and using this equation, we find that

$$[x, z]yd(x) \in P, \quad \text{for all } x, y, z \in R.$$

By the same methods after (3) in the proof of Theorem 1, we obtain that $d(R) \subseteq P$ or R/P is commutative integral domain.

Theorem 2.11 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F\left([x, y]\right) \pm x^m \left[x, y\right] x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. By the hypothesis, we have

$$F\left([x, y]\right) \pm x^m \left[x, y\right] x^n \in P, \quad \text{for all } x, y \in R. \tag{17}$$

Replacing y by yx in this equation, we get

$$F\left([x, y]\, x\right) \pm x^m \left[x, y\right] x^{n+1} \in P$$

and so

$$F\left([x, y]\right)x + [x, y]\, d(x) \pm x^m \left[x, y\right] x^{n+1} \in P, \quad \text{for all } x, y \in R.$$

Using the hypothesis, we obtain that

$$[x, y]\, d(x) \in P, \quad \text{for all } x, y \in R.$$

Taking $zy, z \in R$ for y in this equation and using this, we have

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Using the same arguments after (3) in the proof of Theorem 1, we get the required result.

Theorem 2.12 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F(x \circ y) \pm x^m(x \circ y)x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. We assume that

$$F(x \circ y) \pm x^m(x \circ y)x^n \in P, \quad \text{for all } x, y \in R. \tag{18}$$

Replacing y by yx in this equation, we obtain

$$F(x \circ yx) \pm x^m(x \circ yx)x^n \in P$$

and so

$$F((x \circ y)x) \pm x^m((x \circ y)x)x^n \in P, \quad \text{for all } x, y \in R.$$

That is

$$F(x \circ y)x + (x \circ y)d(x) \pm x^m(x \circ y)x^{n+1} \in P.$$

Using the hypothesis, we get

$$(x \circ y)d(x) \in P, \quad \text{for all } x, y \in R.$$

Substituting $yz, z \in R$ for y in this equation and using this, we find that

$$[x, z]Rd(x) \subseteq P, \quad \text{for all } x, z \in R.$$

Arguing the same methods after (3) in the proof of Theorem 1, we obtain the required result.

Theorem 2.13 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F([x, y]) = \pm x^m(xoy)x^n \in P$, for all $x, y \in R, m, n \in \mathbb{Z}$, then $d(R) \subseteq P$ or R/P is commutative integral domain.*

Proof. Let assume that

$$F([x, y]) \pm x^m(xoy)x^n \in P, \quad \text{for all } x, y \in R. \tag{19}$$

Writing y by yx in (19) and using this equation, we arrive that

$$F([x, yx]) \pm x^m(x \circ yx)x^n \in P$$

and so

$$F([x, y]x) \pm x^m((x \circ y)x)x^n \in P, \quad \text{for all } x, y \in R.$$

Hence we get

$$F\left([x,y]x+[x,y]d(x)\pm x^{m}(x\circ y)x^{n+1}\in P.\right.$$

Using the hypothesis, we have

$$[x,y]d(x)\in P,\quad \text{for all } x,y\in R. \tag{20}$$

Taking $zy, z\in R$ for y in this equation and using this, we have

$$[x,z]Rd(x)\subseteq P,\quad \text{for all } x,z\in R.$$

This equation is same as (3) in the proof of Theorem 1. Arguing the same arguments therein, we get the required result.

Theorem 2.14 *Let R be a ring, P a prime ideal of R and F be a generalized derivation associated with a derivation d of R. If $F(xoy)\pm x^{m}\left[x,y\right]x^{n}\in P$, for all $x,y\in R, m,n\in\mathbb{Z}$, then $d(R)\subseteq P$ or R/P is commutative integral domain.*

Proof. Let assume that

$$F(xoy)=\pm x^{m}\left[x,y\right])xx^{n},\quad \text{for all } x,y\in R. \tag{21}$$

Writing y by yx in (21) and using this equation, we arrive that

$$F\left(x\circ yx\right)\pm x^{m}[x,yx]x^{n}\in P$$

and so

$$F\left((x\circ y)x\right)\pm x^{m}([x,y]x)x^{n}\in P,\quad \text{for all } x,y\in R.$$

That is

$$F\left(x\circ y\right)x+(x\circ y)d(x)\pm x^{m}[x,y]x^{n+1}\in P.$$

Using the hypothesis, we get

$$(x\circ y)d(x)\in P,\quad \text{for all } x,y\in R.$$

Substituting $yz, z\in R$ for y in this equation and using this, we find that

$$[x,z]Rd(x)\subseteq P,\quad \text{for all } x,z\in R.$$

By the same arguments after the equation (3) in the proof of Theorem 1, we obtain the required result.

3 Open Problem

The open problem here is to get an answer to the question; Can we get the same results in this paper when F is an semiderivation associated with a nonzero epimorphism g of R? The another open problem is; How to generalize lemmas and theorems provided in this paper for a nonzero ideal or a square-closed Lie ideal of a prime ring?

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