

On a class of Kählerian manifolds

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Abstract

The unit exact Kähler manifolds are introduced as Kähler manifolds with complex distribution of codimension two, whose Kähler form Ω is exact. We study their essential examples as well as their fundamental properties and we give a concrete examples. Secondly, we investigate some symmetric properties on this kind of manifolds.

Keywords: *Kählerian manifolds, Sasakian manifold, semisymmetric, holomorphic pseudosymmetric.*

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1 Introduction

Let (M^n, g) be a Riemannian manifold. The Lie algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}(M)$ and T_pM will stand for the tangent space to M at any point $p \in M$. We denote by R and S the Riemannian tensor and the Ricci tensor respectively defined for all $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad (1)$$

$$\begin{aligned} S(X, Y) &= R(e_i, X, Y, e_i) \\ &= \left(\sum_{i=1}^n g(R(e_i, X)Y, e_i) \right), \end{aligned} \quad (2)$$

where $\{e_i/i = 1, n\}$ is a local orthonormal basis.

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (M^{2n}, J, g) we thus have

$$J^2 = -I, \quad g(JX, JY) = g(X, Y).$$

where I is the identity transformation.

An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]. \quad (3)$$

For an almost Hermitian manifold (M, J, g) , we define the fundamental Kähler form Ω as:

$$\Omega(X, Y) = g(X, JY).$$

(M, J, g) is then called almost Kähler if Ω is closed i.e. $d\Omega = 0$ where d denotes the exterior derivative. It can be shown that this condition for (M, J, g) to be almost Kähler is equivalent to

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0.$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\Omega = 0$ and $N \equiv 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

On a Kählerian manifold, the holomorphic curvature of any J -invariant 2-plane $\pi \subset T_p M$, for all $p \in M$ is defined by ([10], p.134)

$$K(\pi) = R(X, JX, X, JX),$$

where $X \in \pi$ and $g(X, X) = 1$. A Kählerian manifold (M, J, g) of constant holomorphic sectional curvature c is known as a complex space form. The curvature tensor of a complex space form is given by [10] :

$$\begin{aligned} R(X, Y)Z &= -\frac{c}{4} \left(g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY \right. \\ &\quad \left. + 2g(X, JY)JZ \right) \\ &= -\frac{c}{4} \left((X \wedge Y)Z + (JX \wedge JY)Z + 2g(X, JY)JZ \right) \\ &= -\frac{c}{4} R^{\mathcal{H}}(X, Y)Z, \end{aligned} \quad (4)$$

where $R^{\mathcal{H}}(U, V)$ an operator defined by

$$R^{\mathcal{H}}(U, V) = U \wedge V + JU \wedge JV + 2g(U, JV)J$$

and

$$X \wedge Y = g(Y, Z)X - g(X, Z)Y.$$

The operators $R(U, V)$ and $R^{\mathcal{H}}(U, V)$ will be treated as derivations of the tensor algebra on M in the usual sense. The $(0, 6)$ -tensors fields $R.S$, $R.R$ and $R^{\mathcal{H}}.R$ are defined by

$$\begin{aligned} (R.S)(X, Y; U, V) &= (R(U, V).S)(X, Y) \\ &= -S(R(U, V)X, Y) - S(X, R(U, V)Y). \end{aligned}$$

$$\begin{aligned} (R.R)(X_1, \dots, X_4; U, V) &= (R(U, V).R)(X_1, \dots, X_4) \\ &= -R(R(U, V)X_1, \dots, X_4) - \dots \\ &\quad \dots - R(X_1, \dots, R(U, V)X_4). \end{aligned}$$

$$\begin{aligned} (R^{\mathcal{H}}.R)(X_1, \dots, X_4; U, V) &= (R^{\mathcal{H}}(U, V).R)(X_1, \dots, X_4) \\ &= -R(R^{\mathcal{H}}(U, V)X_1, \dots, X_4) - \dots \\ &\quad \dots - R(X_1, \dots, R^{\mathcal{H}}(U, V)X_4). \end{aligned}$$

A Riemannian manifold (M, g) is said to be Ricci semisymmetric (resp., semisymmetric) if

$$R.S = 0, \quad (\text{resp.}, \quad R.R = 0).$$

A Kählerian manifold (M, J, g) is Ricci holomorphically pseudosymmetric (resp., holomorphically pseudosymmetric) if and only if there exist a function f on M such that

$$R.S = f R^{\mathcal{H}}.S, \quad (\text{resp.}, \quad R.R = f R^{\mathcal{H}}.R).$$

It is clear that any Ricci semisymmetric Kählerian manifold is Ricci holomorphically pseudosymmetric and any semisymmetric Kählerian manifold is holomorphically pseudosymmetric (for $f = 0$).

A holomorphic pseudosymmetric was defined by Olszak [7] and studied by Hotloś [3], Jelonek [4] and others.

2 Preliminaries

2.1 Definition and examples

On a Kählerian manifold of real dimension $2n$, we can assume the existence of a global vector field ξ on M . We denote by η and $\tilde{\eta}$ the 1-forms corresponding to ξ and $J\xi$, respectively, i.e $\forall X \in \mathfrak{X}(M)$

$$\eta(X) = g(\xi, X), \quad \tilde{\eta}(X) = g(\tilde{\xi}, X) = -\eta(JX).$$

Denote by \mathcal{D} the J -invariant distribution of $\dim \mathcal{D} = 2n$ defined by

$$\mathcal{D} = \{X \in T_p M : \eta(X) = \tilde{\eta}(X) = 0\}.$$

In the presence of the distribution \mathcal{D} the structure of any tangent space is $T_p M = \mathcal{D}(p) \oplus \mathcal{D}^\perp(p)$, where $\mathcal{D}^\perp(p)$ is the 2-dimensional J -invariant orthogonal complement to the space $\mathcal{D}(p)$. This means that the structural group of the manifolds under consideration is the subgroup $U(n-1) \times U(1)$ of $U(n)$.

Definition 2.1 *Let (M, J, g) be a Kählerian manifold and ξ a unit global vector field on M where $\eta = \xi^\flat$ its metric dual with respect to g and $\tilde{\eta} = (J\xi)^\flat$. Then, (M, J, g) is said to be a unit exact Kähler manifold if*

$$d\eta = \Omega \quad \text{and} \quad d\tilde{\eta} = 0. \quad (5)$$

Briefly, we will denote such a manifold by (M, J, η, g) . To prove the existence of this kind of manifolds, we will present a concrete examples (2.1) and (3.2). Also, based on Blair's technique [2] we give a method to construct a unit exact Kählerian manifold starting from a Sasakian manifold.

Example 2.1 *For this example, we rely on the example of Z. Olszak [6]. Let (x, y, z, t) denote the Cartesian coordinates in \mathbb{R}^4 . Let (θ^i) be the dual frame of differential 1-forms on \mathbb{R}^4 given by*

$$\theta^1 = e^t dx, \quad \theta^2 = e^t dy, \quad \theta^3 = e^{2t}(-2x dy + dz), \quad \theta^4 = dt,$$

and let (e_i) be the frame of vector fields on M defined by:

$$e_1 = e^{-t} \frac{\partial}{\partial x}, \quad e_2 = e^{-t} \left(\frac{\partial}{\partial y} + 2x \frac{\partial}{\partial z} \right), \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

For the non-zero Lie brackets of (e_i) , we have:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = 2e_3.$$

Define an almost complex structure J on M by assuming

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3$$

For the Nijenhuis tensor N_j , it can be checked that

$$N_j(e_i, e_j) = [Je_i, Je_j] - J[e_i, Je_j] - J[Je_i, e_j] + J^2[e_i, e_j] = 0,$$

for any i, j . By the Newlander-Nirenberg theorem, J is a complex structure on M .

Let g be the Riemannian metric on M for which (e_i) is an orthonormal frame,

so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that the pair (J, g) is a Hermitian structure on M . For the fundamental form Ω , $\Omega(X, Y) = g(X, JY)$, we have

$$\begin{aligned}\Omega &= -2(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) \\ &= -2e^{2t}(dx \wedge dy + dz \wedge dt - 2xdy \wedge dt) \\ &= d\left(e^{2t}(-2xdy + dz)\right).\end{aligned}$$

Putting

$$\eta = e^{2t}(-2xdy + dz) = \theta^3,$$

we get

$$\xi = e_3, \quad \tilde{\xi} = e_4, \quad \tilde{\eta} = \theta^4.$$

Hence Ω is exact, i.e., $d\Omega = 0$. Thus, the pair (J, g) becomes a Kählerian structure on M . Notice that

$$d\eta = \Omega, \quad d\tilde{\eta} = 0, \quad \text{and} \quad \eta(\xi) = 1.$$

Thus, $(\mathbb{R}^4, J, \eta, g)$ is a unit exact Kählerian manifold.

2.2 From Sasakian manifold to unit exact Kählerian manifold

The notion of Sasakian manifold is very well known. First, we recall some results concerning this manifold.

An odd-dimensional Riemannian manifold (M^{2n+1}, g_0) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ tensor field φ , a vector field ξ_0 and a 1-form η_0 such that

$$\eta_0(\xi_0) = 1, \quad \varphi^2(X) = -X + \eta_0(X)\xi_0 \text{ and } g(\varphi X, \varphi Y) = g(X, Y) - \eta_0(X)\eta_0(Y),$$

for any vector fields X, Y on M . In particular, in an almost contact metric manifold we also have $\varphi\xi_0 = 0$ and $\eta_0 \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta_0 = \phi$, where $\phi(X, Y) = g_0(X, \varphi Y)$ is called the fundamental 2-form of M . On the other hand, the almost contact metric structure of M is said to be normal if $[\varphi, \varphi](X, Y) = -2d\eta_0(X, Y)\xi_0$ where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ , given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g_0(X, Y)\xi_0 - \eta_0(Y)X.$$

For more details see for example [1] and [10].

Let $(M^{2n+1}, \varphi, \xi_0, \eta_0, g_0)$ be a Sasakian manifold. On the product manifold $\tilde{M} = \mathbb{R} \times M$ we define a Riemannian metric by

$$g = dt^2 + e^{2t}g_0 + e^{2t}(e^{2t} - 1)\eta_0 \otimes \eta_0, \quad (6)$$

and an almost complex structure J by

$$J(a\partial_t, X) = (e^{2t}\eta_0(X)\partial_t, \varphi X - ae^{-2t}\xi_0), \quad (7)$$

For brevity, we have designated the unit tangent field to \mathbb{R} by ∂_t .

According to Blair [2], (\tilde{M}, J, g) is a Kählerian manifold. Then, the fundamental 2-form Ω of (J, g) is

$$\begin{aligned} \Omega((a\partial_t, X), (b\partial_t, Y)) &= g((a\partial_t, X), J(b\partial_t, Y)) \\ &= e^{2t}(2dt \wedge \eta_0 + \phi)(X, Y). \end{aligned} \quad (8)$$

Since $(\varphi, \xi_0, \eta_0, g_0)$ is a Sasakian structure then $\phi = d\eta_0$ which gives

$$\begin{aligned} \Omega &= e^{2t}(2dt \wedge \eta_0 + d\eta_0) \\ &= d(e^{2t}\eta_0). \end{aligned} \quad (9)$$

Putting $\eta = e^{2t}\eta_0$ we get $\xi = e^{-2t}\xi_0$. Knowing that $\eta_0(\xi_0) = 1$ then $\eta(\xi) = 1$. On the other hand, we have

$$\begin{aligned} \tilde{\eta}(\partial_t) &= -\eta(J\partial_t) = e^{2t}\eta_0(J\partial_t) \\ &= 1 \end{aligned}$$

and for all $X \in \mathfrak{X}(M)$

$$\begin{aligned} \tilde{\eta}(X) &= -\eta(JX) = -e^{2t}\eta_0(JX) \\ &= -\eta(\varphi X + e^{2t}\eta_0(X)\partial_t) \\ &= 0, \end{aligned}$$

implies $\tilde{\eta} = dt$ then $d\tilde{\eta} = 0$. So, (\tilde{M}, J, g, η) is a unit exact Kählerian manifold.

Remark 2.2 *Following the construction above, we notice that there is a one-to-one correspondence between Sasakian and unit exact Kählerian structures.*

3 Main results

3.1 Curvature and main results

Lemma 3.1 *Let (M, J, g, η) be a unit exact Kählerian manifold. For all $X \in \mathfrak{X}(M)$, we have:*

- 1) $\nabla_X \xi = -J(\mathcal{L}_\xi J)X$.
- 2) $\nabla_{JX} \xi + J\nabla_X \xi = 2X$.
- 3) $\nabla_\xi \xi = -2\tilde{\xi}$.

proof 3.2 For the first formula, it suffices to use the definitions of the Lie derivative considering that $\nabla J = 0$. For the second, for all $X, Y \in \mathfrak{X}(M)$ we have

$$\Omega(X, Y) = d\eta(X, Y) \Leftrightarrow 2g(X, JY) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X). \quad (10)$$

Taking $Y = JX$ we get

$$2g(X, X) = g(J\nabla_X \xi, X) + g(\nabla_{JX} \xi, X),$$

then $J\nabla_X \xi + \nabla_{JX} \xi = 2X$.

For the last formula, using equation (10) with $X = \xi$ and knowing that $g(\nabla_Y \xi, \xi) = 0$ we get

$$\nabla_\xi \xi = -2J\xi = -2\tilde{\xi}.$$

One can ask if it is possible to find an explicit expression to $\nabla_X \xi$ for all $X \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of a Riemannian metric. For the answer, assuming

$$\nabla_X \xi = aX + bJX + k\eta(X)\xi + l\eta(X)\tilde{\xi} + p\tilde{\eta}(X)\xi + q\tilde{\eta}(X)\tilde{\xi}, \quad (11)$$

where $a, b, k, l, p, q \in \mathbb{R}$.

First, we have for all $X \in \mathfrak{X}(M)$

$$g(\nabla_X \xi, \xi) = 0 \Leftrightarrow (a + k)\xi - (b - p)J\xi = 0,$$

which give

$$k = -a \quad \text{and} \quad p = b, \quad (12)$$

Knowing that $\nabla_\xi \xi = -2\tilde{\xi}$ (see lemma 3.1), we get

$$l = -b - 2, \quad (13)$$

since $d\tilde{\eta} = 0$ then

$$\nabla_{\tilde{\xi}} \tilde{\xi} = 0 \Leftrightarrow \nabla_{\tilde{\xi}} \xi = 0,$$

replacing in (11) we get

$$q = -a, \quad (14)$$

replacing (12), (13) and (14) in (11) we obtain

$$\nabla_X \xi = aX + bJX - a\eta(X)\xi - (2 + b)\eta(X)\tilde{\xi} + b\tilde{\eta}(X)\xi - a\tilde{\eta}(X)\tilde{\xi}. \quad (15)$$

On the other hand, using (15) in the equation

$$2d\tilde{\eta}(X, Y) = 0 \Leftrightarrow g(\nabla_X \xi, JY) = g(\nabla_Y \xi, JX),$$

we get $a = 0$, so

$$\nabla_X \xi = bJX - (2 + b)(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi), \quad (16)$$

using (16) in (10) we have

$$g(X, JY) = g\left(-bX + (1 + b)(\eta(X)\xi + \tilde{\eta}(X)\tilde{\xi}), JY\right),$$

which give

$$(1 + b)(-X + \eta(X)\xi + \tilde{\eta}(X)\tilde{\xi}) = 0,$$

which implies that $b = -1$ finally, we find

$$\nabla_X \xi = -JX - \eta(X)\tilde{\xi} - \tilde{\eta}(X)\xi.$$

Thus we have

Theorem 3.3 *Let (M, J, g) be a $(2n+2)$ -dimensional Kählerian manifold and ξ a unit global vector field on M . We denote by η and $\tilde{\eta}$ the 1-forms corresponding to ξ and $\tilde{\xi} = J\xi$, respectively. Then, (M, J, η, g) is a unit exact Kählerian manifold if and only if for all $X \in \mathfrak{X}(M)$*

$$\nabla_X \xi = -JX - \eta(X)\tilde{\xi} - \tilde{\eta}(X)\xi. \quad (17)$$

proof 3.4 *The necessity was observed above. For the sufficiency, we suppose that*

$$\nabla_X \xi = -JX - \eta(X)\tilde{\xi} - \tilde{\eta}(X)\xi. \quad (18)$$

Then, using (18), we have

$$\begin{aligned} 2d\eta(X, Y) &= g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) \\ &= 2g(X, JY) \\ &= 2\Omega(X, Y), \end{aligned}$$

and

$$\begin{aligned} 2d\tilde{\eta}(X, Y) &= g(\nabla_X \tilde{\xi}, Y) - g(\nabla_Y \tilde{\xi}, X) \\ &= -g(\nabla_X \xi, JY) + g(\nabla_Y \xi, JX) \\ &= 0. \end{aligned}$$

which completes the proof.

Remark 3.5 *The vector field ξ is a K-torse forming vector field (see [8] and [9]).*

Example 3.1 *Using the above example (2.1). For the Levi-Civita connection corresponding to g , we have*

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_4, & \nabla_{e_1} e_2 &= e_3, & \nabla_{e_1} e_3 &= -e_2, & \nabla_{e_1} e_4 &= e_1, \\ \nabla_{e_2} e_1 &= -e_3, & \nabla_{e_2} e_2 &= -e_4, & \nabla_{e_2} e_3 &= e_1, & \nabla_{e_2} e_4 &= e_2, \\ \nabla_{e_3} e_1 &= -e_2, & \nabla_{e_3} e_2 &= e_1, & \nabla_{e_3} e_3 &= -2e_4, & \nabla_{e_3} e_4 &= 2e_3, \\ \nabla_{e_4} e_1 &= \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0. \end{aligned}$$

We can easily check that for all $X \in \mathfrak{X}(M)$

$$\nabla_X \xi = -JX - \eta(X)\tilde{\xi} - \tilde{\eta}(X)\xi,$$

i.e. for all $i \in \{1, \dots, 4\}$,

$$\nabla_{e_i} e_3 = -J e_i - \theta^3(e_i)e_4 - \theta^4(e_i)e_3.$$

The famous equation (17) gives important informations about the curvature properties of the unit exact Kähler manifold. We start with the first proposition:

Proposition 3.6 *Let (M, J, g, η) be a unit exact Kählerian manifold. Then we have:*

$$R(X, Y)\xi = -(X \wedge Y)\xi - (JX \wedge JY)\xi - 2g(X, JY)J\xi, \quad (19)$$

$$R(X, \xi)Y = -(X \wedge \xi)Y - (JX \wedge J\xi)Y - 2\tilde{\eta}(X)JY, \quad (20)$$

$$S(X, \xi) = -2(n+2)\eta(X), \quad (21)$$

where R denote the curvature tensor and S is the Ricci curvature defined in (1) and (2) respectively.

proof 3.7 *The relation (19) follows from (1) with $Z = \xi$ and formula (17). For the second relation, we have for all vectors fields X, Y, Z on M*

$$g(R(X, \xi)Y, Z) = -g(R(Y, Z)\xi, X),$$

and using (19). Finally, knowing that

$$S(X, Y) = \sum_{i=1}^{2n+2} g(R(X, e_i)e_i, Y),$$

then

$$S(X, \xi) = - \sum_{i=1}^{2n+2} g(R(X, e_i)\xi, e_i),$$

using (19) we obtain (21). This completes the proof of the proposition.

Notice that, from equation (19) we can get

$$R(\xi, J\xi, \xi, J\xi) = 4,$$

then the holomorphic sectional curvature of the plane section generated by $\{\xi, J\xi\}$ is -4 . In addition, if M is a unit exact Kählerian manifold of constant holomorphic sectional curvature c , i.e.

$$R(X, Y)Z = -\frac{c}{4}R^{\mathcal{H}}(X, Y)Z,$$

putting $Z = \xi$ and using formula (19) we get $c = 4$. Thus we have

Proposition 3.8 *If M is a unit exact Kählerian manifold of constant holomorphic sectional curvature c then $c = 4$.*

On the other hand, Let X be the unit vector field orthogonal to ξ and $\tilde{\xi}$. Then, by use of (19), we obtain $R(X, \xi)\xi = -X$. Therefore

$$R(X, \xi, \xi, X) = g(R(X, \xi)\xi, X) = -1.$$

Thus we have

Proposition 3.9 *On a unit exact Kählerian manifold, the sectional curvature of all plane section containing ξ and $\tilde{\xi}$ is -1 .*

3.2 Some symmetric properties on a unit exact Kählerian manifolds

While the unit exact Kählerian manifolds is an area of possible future research, we study in this section some symmetric properties on such class of manifolds.

3.2.1 Ricci Semisymmetric:

Ricci semisymmetric unit exact Kählerian manifold, that is, the manifold satisfying the condition

$$R(U, V).S = 0.$$

Then, we have for all $X, Y, U, V \in \mathfrak{X}(M)$

$$(R(U, V).S)(X, Y) = -S(R(U, V)X, Y) - S(X, R(U, V)Y) \quad (22)$$

Putting $U = X = \xi$ in equation (22) and using (19) and (21), we get

$$(R(\xi, V).S)(\xi, Y) = S(V, Y) + 2(n+2)g(V, Y). \quad (23)$$

By the above discussions we have the following:

Proposition 3.10 *Every Ricci semisymmetric unit exact Kählerian manifold is an Einstein manifold of the form*

$$S(Y, V) = -2(n+2)g(Y, V). \quad (24)$$

3.2.2 Holomorphically Ricci pseudosymmetric:

In this paragraph we study Ricci holomorphically pseudosymmetric unit exact Kählerian manifold, that is, the manifold satisfying the condition

$$R(U, V).S = f R^{\mathcal{H}}(U, V).S.$$

Assume that M is a Ricci holomorphically pseudosymmetric unit exact Kählerian manifold and $X, Y, U, V \in \mathfrak{X}(M)$. We have

$$(R(U, V).S)(X, Y) = f (R^{\mathcal{H}}(U, V).S)(X, Y). \quad (25)$$

Substituting $U = \xi$ in (25) we get

$$(R(\xi, V).S)(X, Y) = f (R^{\mathcal{H}}(\xi, V).S)(X, Y). \quad (26)$$

From formula (20), one can see that

$$R^{\mathcal{H}}(\xi, V) = -R(\xi, V),$$

so, the formula (26) yields

$$(1 + f)R(\xi, V).S = 0. \quad (27)$$

Then, either $f = -1$ or $R(\xi, V).S = 0$.

If M is a non Ricci semisymmetric ($R.S \neq 0$) Ricci holomorphically pseudosymmetric unit exact Kählerian manifold then $R(\xi, V).S \neq 0$. We then have the following Theorem.

Theorem 3.11 *Every essentially Ricci holomorphically pseudosymmetric (that is, different from Ricci semisymmetric) unit exact Kählerian manifold is of the form*

$$R.S = -R^{\mathcal{H}}.S.$$

Conversely, if the manifold satisfies $R(\xi, V).S = 0$, then from formula (23), we have $S = -2(n + 2)g$. Since $R.g = 0$ therefore, it is clear that $R.S = 0$. Then we have:

Theorem 3.12 *A unit exact Kählerian manifold is Ricci holomorphically pseudosymmetric if and only if the $R(\xi, V).S = 0$ and $f = -1$.*

3.2.3 Semisymmetric:

Semisymmetric unit exact Kählerian manifold, that is, the manifold satisfying the condition

$$R(U, V).R = 0.$$

First, we state the following proposition

Proposition 3.13 *For Every unit exact Kählerian manifold, we have*

$$(R(\xi, V).R)(\xi, Y, Z, W) = R(V, Y, Z, W) + g\left(R^{\mathcal{H}}(V, Y)Z, W\right). \quad (28)$$

proof 3.14 *By a long but straight forward computations, using the definition of $R.R$, one can easily prove it.*

Assume that M is a semisymmetric unit exact Kählerian manifold and $X, Y, Z, W, U, V \in \mathfrak{X}(M)$. We have

$$(R(U, V).R)(X, Y, Z, W) = 0. \quad (29)$$

Substituting $X = U = \xi$ and using Proposition (3.13), we get

$$R(V, Y)Z = -R^{\mathcal{H}}(V, Y)Z, \quad (30)$$

shows that the manifold is of constant holomorphic sectional curvature. Thus we state the following result.

Theorem 3.15 *A semisymmetric unit exact Kählerian manifold is of constant holomorphic sectional curvature $c = 4$.*

Let $\{e_i\}$, $i \in \{1, \dots, 2n + 2\}$ be an orthonormal basis of the tangent space at any point. Now taking summation over $i = 1, 2, \dots, 2n + 2$ of the relation (30) for $Y = Z = e_i$ gives

$$S(V, Y) + 2(n + 2)g(V, Y) = 0. \quad (31)$$

Then, the manifold is an Einstein manifold of the form

$$S(V, Y) = -2(n + 2)g(V, Y). \quad (32)$$

By the above discussions we have the following:

Proposition 3.16 *Every semisymmetric unit exact Kählerian manifold is an Einstein manifold of the form (32).*

Example 3.2 *in [7], Olszak gave an important example of a 1-parameter family of Kählerian manifolds. We will use it here, with giving parameter value 1.*

Let $(x^\alpha, y^\alpha, z, t)$ denote the Cartesian coordinates in \mathbb{R}^{2m+2} , $m \geq 1$. Latin indices take on values from 1 to $2m + 2$, greek indices will run from 1 to m and $\alpha' = \alpha + m$ for all $\alpha \in \{1, \dots, m\}$.

Assume that $M = N \times (A, B) \subset \mathbb{R}^{2m+2}$ where N is an open connected subset of \mathbb{R}^{2m+1} and (A, B) is an open interval and $B > A > 0$. Suppose that $h : (A, B) \rightarrow \mathbb{R}$ is a smooth function which non-zero at any $t \in (A, B)$.

Let (e_i) be the frame of vector fields on M defined by:

$$e_\alpha = \frac{1}{t} \frac{\partial}{\partial x^\alpha}, \quad e_{\alpha'} = \frac{1}{t} \left(\frac{\partial}{\partial y^\alpha} + 2x^\alpha \frac{\partial}{\partial z} \right), \quad e_{2m+1} = \frac{1}{t^2 h} \frac{\partial}{\partial z}, \quad e_{2m+2} = th \frac{\partial}{\partial t},$$

and let (θ^i) be the dual frame of differential 1-forms,

$$\theta^\alpha = t dx^\alpha, \quad \theta^{\alpha'} = t dy^\alpha, \quad \theta^{2m+1} = t^2 h (-2 \sum_\lambda x^\lambda dy^\lambda + dz), \quad \theta^{2m+2} = \frac{1}{th} dt.$$

For the non-zero Lie brackets of (e_i) , we have:

$$[e_\alpha, e_{\beta'}] = 2h \delta_{\alpha\beta} e_{2m+1}, \quad [e_\alpha, e_{2m+2}] = h e_\alpha,$$

$$[e_{\alpha'}, e_{2m+2}] = h e_{\alpha'}, \quad [e_{2m+1}, e_{2m+2}] = (2h + th') e_{2m+1}.$$

Define an almost complex structure J on M by assuming

$$J e_\alpha = e_{\alpha'}, \quad J e_{\alpha'} = -e_\alpha, \quad J e_{2m+1} = e_{2m+2}, \quad J e_{2m+2} = -e_{2m+1}$$

For the Nijenhuis tensor N_j , it can be checked that

$$N_j(e_i, e_j) = [J e_i, J e_j] - J[e_i, J e_j] - J[J e_i, e_j] + J^2[e_i, e_j] = 0,$$

for any i, j . By the Newlander-Nirenberg theorem, J is a complex structure on M .

Let g be the Riemannian metric on M for which (e_i) is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that the pair (J, g) is a Hermitian structure on M . For the fundamental form Ω , $\Omega(X, Y) = g(X, JY)$, we have

$$\begin{aligned} \Omega &= -2 \sum_\lambda \theta^\lambda \wedge \theta^{\bar{\lambda}} - 2\theta^{2m+1} \wedge \theta^{2m+2} \\ &= -2t^2 \sum_\lambda dx^\lambda \wedge dy^\lambda - 2t \left(\sum_\lambda x^\lambda dy^\lambda \wedge dt + dz \wedge dt \right) \\ &= d \left(t^2 \left(-2 \sum_\lambda x_\lambda dy^\lambda + dz \right) \right), \end{aligned}$$

Putting

$$\eta = t^2 \left(-2 \sum_\lambda x_\lambda dy^\lambda + dz \right) = \frac{1}{h} \theta^{2m+1},$$

let's take $h = 1$ we get

$$\eta = \theta^{2m+1}, \quad \xi = e_{2m+1}, \quad \tilde{\xi} = e_{2m+2}, \quad \tilde{\eta} = \theta^{2m+2}.$$

Hence Ω is exact, i.e., $d\Omega = 0$. Thus, the pair (J, g) becomes a Kählerian structure on M . Since $\eta(\xi) = 1$ and $d\tilde{\eta} = 0$ then (M, J, η, g) is a unit exact Kählerian manifold.

For the Levi-Civita connection corresponding to g , we have

We can easily check that for all $X \in \mathfrak{X}(M)$

$$\nabla_X \xi = -JX - \eta(X) \tilde{\xi} - \tilde{\eta}(X) \xi,$$

$$\begin{aligned}
\nabla_{e_\alpha} e_\beta &= \nabla_{e_{\alpha'}} e_{\beta'} = -\delta_{\alpha\beta} e_{2m+2}, \\
\nabla_{e_\alpha} e_{\beta'} &= -\nabla_{e_{\alpha'}} e_\beta = \delta_{\alpha\beta} e_{2m+1}, \\
\nabla_{e_\alpha} e_{2m+1} &= \nabla_{e_{2m+1}} e_\alpha = -\nabla_{e_{\alpha'}} e_{2m+2} = -e_{\alpha'}, \\
\nabla_{e_\alpha} e_{2m+2} &= \nabla_{e_{\alpha'}} e_{2m+1} = \nabla_{e_{2m+1}} e_{\alpha'} = e_\alpha, \\
\nabla_{e_{2m+1}} e_{2m+1} &= -2e_{2m+2}, \\
\nabla_{e_{2m+1}} e_{2m+2} &= 2e_{2m+1}.
\end{aligned}$$

i.e. for all $i \in \{2n+2\}$,

$$\nabla_{e_i} \xi = -Je_i - \theta^{2m+1}(e_i)e_{2m+2} - \theta^{2m+2}(Je_i)e_{2m+1}.$$

Let $R_{ijkl}(= R_{ijkl}^l)$ be the components of the curvature tensor R with respect to the adapted frame, $R(e_i, e_j)e_k = \sum_l R_{ijkl}e_l$. The non zero components of R are related to the following:

$$\begin{aligned}
R_{\alpha\beta\gamma\lambda} &= R_{\alpha\beta\gamma'\lambda'} = R_{\alpha'\beta'\gamma'\lambda'} = \delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\alpha\lambda}\delta_{\beta\gamma}, \\
R_{\alpha\beta'\gamma\lambda'} &= \delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\beta\gamma}\delta_{\alpha\lambda} + 2\delta_{\alpha\beta}\delta_{\gamma\lambda}, \\
R_{\alpha\beta'(2m+1)(2m+2)} &= 2\delta_{\alpha\beta}, \\
R_{\alpha(2m+1)\beta(2m+1)} &= R_{\alpha(2m+1)\beta'(2m+2)} \\
&= R_{\alpha(2m+2)\beta(2m+2)} = -R_{\alpha(2m+2)\beta'(2m+1)} \\
&= R_{\alpha'(2m+1)\beta'(2m+1)} = -R_{\alpha'(2m+2)\beta'(2m+2)} \\
&= \delta_{\alpha\beta}, \\
R_{(2m+1)(2m+2)(2m+1)(2m+2)} &= 4.
\end{aligned}$$

Next, we check that all components

$$\begin{aligned}
(R.R)_{pqhijk} &= (R(e_p, e_q).R)_{hijk} = (R(e_p, e_q).R)(e_h, e_i, e_j, e_k) \\
&= -R(R(e_p, e_q)e_h, e_i, e_j, e_k) - R(e_h, R(e_p, e_q)e_i, e_j, e_k) \\
&\quad - R(e_h, e_i, R(e_p, e_q)e_j, e_k) - R(e_h, e_i, e_j, R(e_p, e_q)e_k) \\
&= -R\left(\sum_s R_{pqhs}e_s, e_i, e_j, e_k\right) - R\left(e_h, \sum_s R_{pqis}e_s, e_j, e_k\right) \\
&\quad - R\left(e_h, e_i, \sum_s R_{pqjs}e_s, e_k\right) - R\left(e_h, e_i, e_j, \sum_s R_{pqks}e_s\right) \\
&= -\sum_s \left(R_{pqhs}R_{sijk} + R_{pqis}R_{hsjk} + R_{pqjs}R_{hisk} + R_{pqks}R_{hij s} \right)
\end{aligned}$$

vanish identically. So, The manifold (M, J, g) is semisymmetric.

For the non-zero components of the Ricci curvature tensor S , we have

$$S_{\alpha\alpha} = S_{\alpha'\alpha'} = S_{(2m+1)(2m+1)} = S_{(2m+2)(2m+2)} = -2(m+2).$$

So, notice that (M, J, η, g) is an Einstein manifold with $S = -2(m+2)g$.

3.2.4 Holomorphically pseudosymmetric:

In this paragraph we study holomorphically pseudosymmetric unit exact Kählerian manifold, that is, the manifold satisfying the condition

$$R(U, V).R = f R^{\mathcal{H}}(U, V).R.$$

Assume that M is an holomorphically pseudosymmetric unit exact Kählerian manifold and $X, Y, Z, W, U, V \in \mathfrak{X}(M)$. We have

$$(R(U, V).R)(X, Y, Z, W) = f (R^{\mathcal{H}}(U, V).R)(X, Y, Z, W). \quad (33)$$

Substituting $U = \xi$ in (33) we get

$$(R(\xi, V).R)(X, Y, Z, W) = f (R^{\mathcal{H}}(\xi, V).R)(X, Y, Z, W). \quad (34)$$

From formula (20), one can see that

$$R^{\mathcal{H}}(\xi, V) = -R(\xi, V),$$

so, the formula (34) yields

$$(1 + f)R(\xi, V).R = 0 \quad (35)$$

Then either $f = -1$ or, $R(\xi, V).R = 0$.

If M is a non semisymmetric ($R.R \neq 0$) holomorphically pseudosymmetric unit exact Kählerian manifold then $R(\xi, V).R \neq 0$. Then, we have the following Theorem.

Theorem 3.17 *Every essentially holomorphically pseudosymmetric (that is, different from semisymmetric) unit exact Kählerian manifold is of the form*

$$R.R = -R^{\mathcal{H}}.R.$$

Conversely, if the manifold satisfies $R(\xi, V).R = 0$, then from proposition (3.13) we get $R = -R^{\mathcal{H}}$ therefore, it is clear that $R.R = -R^{\mathcal{H}}.R$. This leads the following:

Theorem 3.18 *A unit exact Kählerian manifold is holomorphically pseudosymmetric if and only if the $R(\xi, V).R = 0$ and $f = -1$.*

Example 3.3 *Based on the Proposition (3.8), If M is a unit exact Kählerian manifold of constant holomorphic sectional curvature c then $c = 4$ and we have $R = -R^{\mathcal{H}}$ which gives $R.R = -R^{\mathcal{H}}.R$. Then, every unit exact Kählerian of constant holomorphic sectional curvature c is holomorphically pseudosymmetric.*

4 Open Problem

It is well known that the class of pseudosymmetric Kähler manifolds is non-essential in the sense that they are necessarily semisymmetric in any dimension greater than 4. So, the open problem here is about pseudosymmetry of 4-dimensional unit exact Kähler manifolds.

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