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T-fuzzy subalgebras of BCI-algebras

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Abstract

In this study, we construct a model for fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCI-algebra X under t-norm T and we investigate the relationships between them and fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCI-algebra. In the Second step we define the intersection and direct product of them such that we prove that the intersection and direct product of them will be also fuzzy ideals and fuzzy positive implicative ideals of BCI-algebra X under t-norm T and we make a theoretical study their basic properties of them. Finally, we investigate the image and pre image of them under homomorphisms of BCI-algebras and we obtain some results about them.

Keywords: *Algebra and orders, theory of fuzzy sets, norms, products and intersections, homomorphisms.*

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1 Introduction

In mathematics, a subalgebra is a subset of an algebra, closed under all its operations, and carrying the induced operations. "Algebra", when referring to a structure, often means a vector space or module equipped with an additional bilinear operation. Algebras in universal algebra are far more general: they are a common generalisation of all algebraic structures. "Subalgebra" can refer to either case. A subalgebra of an algebra over a commutative ring or field is a vector subspace which is closed under the multiplication of vectors. The restriction of the algebra multiplication makes it an algebra over the same ring or field. This notion also applies to most specializations, where the multiplication must satisfy additional properties, e.g. to associative algebras or to Lie algebras. Only for unital algebras is there a stronger notion, of unital subalgebra, for which it is also required that the unit of the subalgebra be the unit of the bigger algebra. In algebra, a homomorphism is a structure-preserving map between two algebraic structures of the same type (such as two groups, two rings, or two vector spaces). The word homomorphism comes from the Ancient Greek language: (homos) meaning "same" and (morphe) meaning "form" or "shape". However, the word was apparently introduced to mathematics due to a (mis)translation of German ähnlich meaning "similar" to meaning "same". The term "homomorphism" appeared as early as 1892, when it was attributed to the German mathematician Felix Klein (1849-1925). Homomorphisms of vector spaces are also called linear maps, and their study is the subject of linear algebra. The concept of homomorphism has been generalized, under the name of morphism, to many other structures that either do not have an underlying set, or are not algebraic. This generalization is the starting point of category theory. A homomorphism may also be an isomorphism, an endomorphism, an automorphism. Each of those can be defined in a way that may be generalized to any class of morphisms. In mathematics, intersection theory is one of the main branches of algebraic geometry, where it gives information about the intersection of two subvarieties of a given variety. The theory for varieties is older, with roots in Bézout's theorem on curves and elimination theory. On the other hand, the topological theory more quickly reached a definitive form. There is yet an ongoing development of intersection theory. Currently the main focus is on: virtual fundamental cycles, quantum intersection rings, Gromov-Witten theory and the extension of intersection theory from schemes to stacks. In mathematics, ideal theory is the theory of ideals in commutative rings; and is the precursor name for the contemporary subject of commutative algebra. The name grew out of the central considerations, such as the Lasker–Noether theorem in algebraic geometry, and the ideal class group in algebraic number theory, of the commutative algebra of the first quarter of the twentieth century. It was used in the influential van der Waerden text on abstract algebra from around 1930. The ideal theory in question had been based on elimination theory, but in line with David Hilbert's taste moved away from algorithmic methods. Gröbner basis theory has now reversed the trend, for computer algebra. The importance of the idea of a module, more general than an ideal, probably led to the perception that ideal theory was too narrow a description. Valuation theory, too, was an important technical extension, and was used by Helmut

Hasse and Oscar Zariski. Bourbaki used commutative algebra; sometimes local algebra is applied to the theory of local rings. Douglas Northcott's 1953 Cambridge Tract Ideal Theory (reissued 2004 under the same title) was one of the final appearances of the name. Imai and Iseki introduced the notion of BCI-algebra [9]. Fuzzy mathematics is the branch of mathematics including fuzzy set theory and fuzzy logic that deals with partial inclusion of elements in a set on a spectrum, as opposed to simple binary "yes" or "no" (0 or 1) inclusion. Undoubtedly the notion of fuzzy set theory initiated by Zadeh [39] in 1965 in a seminal paper, plays the central role for further development. His seminal paper in 1965 has opened up new insights and applications in a wide range of scientific fields. Azriel Rosenfeld used the notion of a fuzzy subset to set down cornerstone papers in several areas of mathematics, among other disciplines. Rosenfeld is the father of fuzzy abstract algebra.. Many authors introduced and investigated some properties of fuzzy mathematics structures [2, 3, 4, 5, 6, 12, 13, 15, 17, 18, 19, 35, 40]. Fuzzy subalgebras theory is a chapter of fuzzy set theory. It is obtained from an interpretation in a multi-valued logic of axioms usually expressing the notion of subalgebra of a given algebraic structure. Many authors introduced and investigated some properties of fuzzy BCI-algebraic structures[10, 11, 14, 36, 37, 41]. The literature on triangular norms suggests that there are families of (continuous) t-norms from the familiar G˜odelian minimum t-norm to the more general Archimedean t-norms. A well-grounded logical system should be based on a left-continuous t-norm. Many discontinuous tnorms can be approximated by continuous t-norms and therefore we focus only to continuous ones. Hohle [8], Alsina et al. [1] defined and investigated the T-norms into fuzzy set theory. Many other researchers have presented various types of t-norms for particular purposes [7, 38]. In previous works [20]-[34], by using norms, we investigated some properties of fuzzy algebraic structures. In this paper, by the use of t-norm T, new kinds of fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCI-algebra are introduced. The fundamental properties of them are presented. Concepts of intersection and direct product of them are introduced and the relationships between them and fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCI-algebra are investigated and characterized. Finally, homomorphisms of BCI-algebras over them will be investigated and obtained fundamental results.

2 Preliminaries

Definition 2.1 *([14]) an algebra* $(X, *, 0)$ *of type* $(2, 0)$ *is called a BCI-algebra if it satisfies the following conditions:*
(1) $((x * y) * (x * z)) * (z * y) = 0$
(2) $(x * (x * y)) * y = 0$

(3) $x * x = 0$
(4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$
(5) $(x * y) * z = (x * z) * y$
(6) $x * 0 = x$
(7) $0 * (x * y) = (0 * x) * (0 * y)$
(8) $0 * (0 * (x * y)) = 0 * (y * x)$
for all $x, y, z \in X$.
In a BCI-algebra, we can define a partial ordering $" \leq "$ *by* $x \leq y$ *if and only if* $x * y = 0$.
(9) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$
(10) $(x * z) * (y * z) \leq x * y$
for all $x, y, z \in X$.

Definition 2.2 *([14]) A non-empty subset* I *of a* BCI*-algebra* X *is called an ideal of* X *if*
(1) $0 \in I$,
(2) $x * y \in I$ *and* $y \in I$ *imply that* $x \in I$ *for all* $x, y \in X$.

Definition 2.3 *([37]) A non-empty subset* I *of a* BCI*-algebra* X *is said to be a positive implicative ideal of* X *if it satisfies:*
(1) $0 \in I$,
(2) $((x * z) * z) * (y * z) \in I$ *and* $y \in I$ *imply* $x * z \in I$,
for all $x, y, z \in X$.

Definition 2.4 *([37]) A non-empty subset* I *of a* BCI*-algebra* X *is called subalgebra of* X *if* $x * y \in I$ *for all* $x, y \in I$.

Definition 2.5 *([14]) Define a mapping* $f : X \to Y$ *of* BCI*-algebras a homomorphism if* $f(x * y) = f(x) * f(y)$, *for all* $x, y \in X$.

Definition 2.6 *([16]) A fuzzy subset of an arbitrary* X, *we mean a function* $\mu : X \to [0, 1]$ *and* $[0, 1]^X$ *for all fuzzy subsets of* X. *We call the set* $\mu_s = \{x \in X : \mu(x) \geq s\}$ *an upper level of* μ *for all* $s \in [0, 1]$.

Definition 2.7 *([16]) Let* $\varphi : X \to Y$ *be a function such that* $\mu : X \to [0, 1]$ *and* $\nu : Y \to [0, 1]$. *We define* $\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$ *and* $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$ *for all* $x \in X$ *and* $y \in Y$.

Definition 2.8 *([16]) Define a t-norm* T *as* $T : [0, 1] \times [0, 1] \to [0, 1]$ *such that :*
(1) $T(x, 1) = x$ *(neutral element),*
(2) $T(x, y) \leq T(x, z)$ *if* $y \leq z$ *(monotonicity),*
(3) $T(x, y) = T(y, x)$ *(commutativity),*
(4) $T(x, T(y, z)) = T(T(x, y), z)$ *(associativity),*
for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.9 *([16]) (1) Standard intersection t-norm* $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum t-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product t-norm $T_p(x, y) = xy$.
(4) Drastic t-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

We say that T be idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$.

Definition 2.10 *([20]) Let* $\mu, \nu : X \to [0, 1]$ *define* $\mu \cap \nu : X \to [0, 1]$ *as* $(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$ *a for all* $x \in X$.

Definition 2.11 *([20]) Let* $\mu, \nu : X \to [0, 1]$ *and* $\nu : Y \to [0, 1]$ *The cartesian product of* μ *and* ν *is denoted by* $\mu \times \nu : X \times Y \to [0, 1]$ *is defined by* $(\mu \times \nu)(x, y) = T(\mu(x), \mu(y))$ *for all* $(x, y) \in X \times Y$.

Lemma 2.12 *([20]) Let* T *be a t-norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3 T-norms over fuzzy subalgebras, ideals and positive implicative ideals of BCI-algebras

Definition 3.1 *Let* $\mu : X \to [0, 1]$ *bea fuzzy subset of* X. *Define* μ *as a fuzzy subalgebra of BCI-algebra* X *under t-norm* T *if* $\mu(x * y) \geq T(\mu(x), \mu(y))$, *for all* $x, y \in X$.
Denote by $FST(X)$, *the set of all fuzzy subalgebras of BCI-algebra* X *under t-norm* T.

Example 3.2 *Let $X = \{0, a, b, c\}$ be a set given by the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X, *, 0)$ is a BCI-algebra.
Define fuzzy subset $\mu : (X, *, 0) \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.85 & \text{if } x = 0, a, c \\ 0.15 & \text{if } x = b \end{cases}$$

let $T(a, b) = T_p(a, b) = ab$ for all a,b $\in [0, 1]$ then $\mu \in FST(X)$.

Proposition 3.3 *Let $\mu \in FST(X)$ and T, C be idempotent. Then $\mu \in FST(X)$ if and only if the set*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

be either empty or a subalgebra of X for every $s \in [0, 1]$.

Proof 3.4 *If $\mu \in FST(X)$ and $x, y \in \mu_s$, then $\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(s, s) = s$ thus $x * y \in \mu_s$ and so μ_s will be a subalgebra of X for every $s \in [0, 1]$.*
Conversely, let $s = T(\mu(x), \mu(y))$ and $x, y \in \mu_s$ and μ_s be a subalgebra of X so $x * y \in A_s$ and then $\mu(x * y) \geq s = T(\mu(x), \mu(y))$ therefore $\mu \in FST(X)$.

Proposition 3.5 *Let $\mu \in FST(X)$ and T be idempotent. Then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof 3.6 *Let $x \in X$. Then*

$$\mu(0) = \mu_A(x * x) \geq T(\mu(x), \mu(x)) = \mu(x).$$

Proposition 3.7 *Let $\mu \in FST(X)$ and $\nu \in FST(X)$. Then $\mu \cap \nu \in FST(X)$.*

Proof 3.8 *Let $x, y \in X$. Then*

$$\begin{aligned} (\mu \cap \nu)(x * y) &= T(\mu(x * y), \nu(x * y)) \\ &\geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(x), (\mu \cap \nu)(y)). \end{aligned}$$

thus

$$(\mu \cap \nu)(x * y) \geq T((\mu \cap \nu)(x), (\mu \cap \nu)(y))$$

so $\mu \cap \nu \in FST(X)$.

Proposition 3.9 *Let $\mu \in FST(X)$ and $\nu \in FST(Y)$. Then $\mu \times \nu \in FST(X \times Y)$.*

Proof 3.10 *Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then*

$$\begin{aligned}(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2)\\ &= T(\mu(x_1 * x_2), \nu(y_1 * y_2))\\ &\geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2)))\\ &= T(T(\mu(x_1), \nu(y_1)), T(\mu(x_2), \nu(y_2)))\\ &= T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))\end{aligned}$$

thus

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) \geq T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)).$$

Therefore $\mu \times \nu \in FST(X \times Y)$.

Proposition 3.11 *If $\mu \in FST(X)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi(\mu) \in FST(Y)$.*

Proof 3.12 *Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Then*

$$\begin{aligned}\varphi(\mu)(y_1 * y_2) &= \sup\{\mu(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ &\geq \sup\{T(\mu(x_1), \mu_(x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ &\geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2)))\\ &= T(\sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \sup\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})\\ &= T(\varphi(\mu)(y_1), \varphi(\mu)(y_2))\end{aligned}$$

thus

$$\varphi(\mu)(y_1 * y_2) \geq T(\varphi(\mu)(y_1), \varphi(\mu)(y_2)).$$

Then $\varphi(\mu) \in FST(Y)$.

Proposition 3.13 *If $\nu \in FST(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(\nu) \in FST(X)$.* Let $x_1, x_2 \in X$. Then

$$\begin{aligned}\varphi^{-1}(\nu)(x_1 * x_2) &= \nu(\varphi(x_1 * x_2))\\ &= \nu(\varphi(x_1) * \varphi(x_2))\\ &\geq T(\nu(\varphi(x_1)), \nu(\varphi(x_2)))\\ &= T(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2))\end{aligned}$$

thus

$$\varphi^{-1}(\nu)(x_1 * x_2) \geq T(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in FST(X)$.

Definition 3.14 *Define $\mu : X \to [0,1]$ is a fuzzy ideal of BCI-algebra X under t-norm T if it satisfies the following inequalities:*
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x) \geq T(\mu(x * y), \mu(y))$,
for all $x, y \in X$.
Denote by $FIT(X)$, the set of all fuzzy ideals of X under t-norm T.

Example 3.15 *Let $X = \{0, a, 1, 2, 3\}$ be a set given by the following Cayley table:*

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	3	2	1	0	3
3	3	3	2	1	0

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} t_0 & \text{if } x = 0, \\ t_1 & \text{if } x = a, \\ t_2 & \text{if } x = 1, 2, 3, \end{cases}$$

with $t_0 > t_1 > t_2$ and $t_i \in [0,1]$. Let $T_m(x, y) = \min\{x, y\}$ for all x,y $\in [0,1]$ then $\mu \in FIT(X)$

Proposition 3.16 *Let $\mu \in FIT(X)$ and T be idempotent. Then $\mu \in FIT(X)$ if and only if the*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

be either empty or an ideal of BCI-algebra X for every $s \in [0,1]$.

Proof 3.17 *Let $\mu \in FIT(X)$ and $x, y \in X$. Then $\mu(0) \geq \mu(x) \geq s$ and then $0 \in \mu_s$. Also let $x * y \in \mu_s$ and $y \in \mu_s$. Then*

$$\mu(x) \geq T(\mu(x * y), \mu(y)) \geq T(s, s) = s$$

thus $x \in \mu_s$. Then μ_s will be an ideal of BCI-algebra X for every $s \in [0,1]$. Conversely, let μ_s be either empty or an ideal of BCI-algebra X for every $s \in [0,1]$. Let $s = T(\mu(x * y), \mu(y))$ with $x * y \in \mu_s$ and $y \in \mu_s$. Then $x \in \mu_s$ thus

$$\mu(x) \geq s = T(\mu(x * y), \mu(y))$$

so $\mu \in FIT(X)$.

Proposition 3.18 *Let $\mu \in FIT(X)$ and $x*y \leq z$. Then $\mu(x) \geq T(\mu(y), \mu(z))$ for all $x, y, z \in X$.*

Proof 3.19 *As $x * y \leq z$ so $(x * y) * z = 0$ for all $x, y, z \in X$. Then*

$$\begin{aligned} \mu(x) &\geq T(\mu(x * y), \mu(y)) \\ &\geq T(T(\mu((x * y) * z), \mu(z)), \mu_A(y)) \\ &= T(T(\mu(0), \mu_A(z)), \mu(y)) \\ &= T(\mu(z), \mu(y)) \\ &= T(\mu(y), \mu(z)) \end{aligned}$$

thus $\mu(x) \geq T(\mu(y), \mu(z))$.

Proposition 3.20 *Let $\mu \in FIT(X)$ and $x \leq y$ for all $x, y \in X$. Then $\mu(x) \geq \mu(y)$.*

Proof 3.21 *Since $x \leq y$ so $x * y = 0$ for all $x, y \in X$. Then*

$$\mu(x) \geq T(\mu(x * y), \mu(y)) = T(\mu(0), \mu(y)) = \mu_A(y).$$

.

Proposition 3.22 *Let $\mu \in FIT(X)$ and $\nu \in FIT(X)$ Then $\mu \cap \nu \in FIT(X)$.*

Proof 3.23 *Let $x, y \in X$. Then*
(1)

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x).$$

(2)

$$\begin{aligned} (\mu \cap \nu)(x) &= T(\mu(x), \nu(x)) \\ &\geq T(T(\mu(x * y), \mu(y)), T(\nu(x * y), \nu(y))) \\ &= T(T(\mu(x * y), \nu(x * y)), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(x * y), (\mu \cap \nu)(y)) \end{aligned}$$

so

$$(\mu \cap \nu)(x) \geq T((\mu \cap \nu)(x * y), (\mu \cap \nu)(y)).$$

Thus $\mu \cap \nu \in FIT(X)$.

Proposition 3.24 *Let* $\mu \in FIT(X)$ *and* $\nu \in FIT(Y)$*. Then* $\mu \times \nu \in FIT(X \times Y)$*.*

Proof 3.25 *Let* $(x, y) \in X \times Y$*. Then*

$$(\mu \times \nu)(0,0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(y)) = (\mu \times \nu)(x, y).$$

Also let $x_i \in X$ *and* $y_i \in Y$ *for* $i = 1, 2$*. Now*

$$\begin{aligned}
(\mu \times \nu)(x_1, y_1) &= T(\mu(x_1), \nu(y_1)) \\
&\geq T(T(\mu(x_1 * x_2), \mu(x_2)), T(\nu(y_1 * y_2), \nu(y_2))) \\
&= T(T(\mu(x_1 * x_2), \nu(y_1 * y_2)), T(\mu(x_2), \mu(y_2))) \\
&= T((\mu \times \nu)(x_1 * x_2, y_1 * y_2), (\mu \times \nu)(x_2, y_2)) \\
&= T((\mu \times \nu)((x_1, y_1) * (x_2, y_2)), (\mu \times \nu)(x_2, y_2))
\end{aligned}$$

thus

$$(\mu \times \nu)(x_1, y_1) \geq T((\mu \times \nu)((x_1, y_1) * (x_2, y_2)), (\mu \times \nu)(x_2, y_2)).$$

Therefore $\mu \times \nu \in FIT(X \times Y)$*.*

Proposition 3.26 *If* $\mu \in FIT(X)$ *and* $\varphi : X \to Y$ *be a homomorphism of* BCI*-algebras, then* $\varphi(\mu) \in FIT(Y)$*.*

Proof 3.27 *Let* $x \in X$ *and* $y \in Y$ *with* $\varphi(x) = y$*. Now*

$$\varphi(\mu)(0) = \sup\{\mu(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y).$$

Also let $x, x_1 \in X$ *such that* $\varphi(x) = y, \varphi(x_1) = y_1$*. Then*

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$$

$$\geq \sup\{T(\mu(x * x_1), \mu(x_1)) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}$$

$$= T(\sup\{\mu(x{*}x_1) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\})$$

$$= T(\sup\{\mu(x{*}x_1) \mid x, x_1 \in X, \varphi(x{*}x_1) = y{*}y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}$$

$$= T(\varphi(\mu)(y * y_1), \varphi(\mu_A)(y_1))$$

therefore

$$\varphi(\mu)(y) \geq T(\varphi(\mu)(y * y_1), \varphi(\mu)(y_1)).$$

Therefore $\varphi(\mu) \in FIT(Y)$*.*

Proposition 3.28 *If $\nu \in FIT(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(\nu) \in FIT(X)$.*

Proof 3.29 *Let $x \in X$. Then*

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \geq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x, x_1 \in X$. As

$$\begin{aligned}\varphi^{-1}(\nu)(x) &= \nu(\varphi(x)) \\ &\geq T(\nu(\varphi(x) * \varphi(x_1)), \nu(\varphi(x_1))) \\ &= T(\nu(\varphi(x * x_1)), \nu(\varphi(x_1))) \\ &= T(\varphi^{-1}(\nu)(x * x_1), \varphi^{-1}(\nu)(x_1))\end{aligned}$$

so

$$\varphi^{-1}(\nu)(x) \geq T(\varphi^{-1}(\nu)(x * x_1), \varphi^{-1}(\mu_B)(x_1)).$$

Therefore $\varphi^{-1}(\nu) \in FIT(X)$.

Definition 3.30 *We say that $\mu : X \to [0,1]$ is a fuzzy positive implicative ideal of BCI-algebra X under t-norm T if it satisfies the following inequalities:*
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x * z) \geq T(\mu(((x * z) * z) * (y * z)), \mu(y))$,
for all $x, y, z \in X$.
Denote by $FPIIT(X)$, the set of all fuzzy positive implicative ideals of BCI-algebra X under t-norm T.

Example 3.31 *Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:*

$*$	*0*	*1*	*2*	*3*
0	*0*	*0*	*0*	*3*
1	*1*	*0*	*0*	*3*
2	*2*	*2*	*0*	*3*
3	*3*	*3*	*3*	*0*

Then $(X, *, 0)$ is a BCI-algebra. Define $\mu : X \to [0,1]$ as

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, 3 \\ t & \text{if } x = 1, 2 \end{cases}$$

such that and $t \in (0,1)$. Let $T_b(x, y) = \max\{0, x+y-1\}$ for all $x, y \in [0,1]$ then $\mu \in FPIIT(X)$.

Proposition 3.32 *Let $\mu : X \to [0,1]$ and T be idempotent. Then $\mu \in FPIIT(X)$ if and only if the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ be either empty or a positive implicative ideal of BCI-algebra X for every $s \in [0,1]$.*

Proof 3.33 *Let* $\mu \in FPIIT(X)$ *and* $\mu_s = \{x \in X : \mu(x) \geq s\}$ *be not empty then for any* $x \in \mu_s$ *we have* $\mu(x) \geq s$ *and thus* $\mu(0) \geq s$ *which means that* $0 \in \mu_s$.
Also let $((x * z) * z) * (y * z) \in \mu_s$ *and* $y \in \mu_s$. *Then*

$$\mu(x * z) \geq T(\mu(((x * z) * z) * (y * z)), \mu(y)) \geq T(s, s) = s$$

thus $x * z \in \mu_s$. *Then* μ_s *is a posive implicative ideal of* X *for every* $s \in [0, 1]$.
Conversely, let μ_s *be not empty and be a positive implicative ideal of* X *for every* $s \in [0, 1]$. *Then for any* $x \in \mu_s$ *we have* $\mu(x) \geq s$. *Let* $s = T(\mu(((x * z) * z) * (y * z)), \mu(y))$ *with* $((x * z) * z) * (y * z) \in \mu_s$ *and* $y \in \mu_s$. *Thus* $x * z \in \mu_s$.
Therefore

$$\mu(x * z) \geq s = T(\mu(((x * z) * z) * (y * z)), \mu(y))$$

so $\mu \in FPIIT(X)$.

Proposition 3.34 *Let* $\mu \in FPIIT(X)$ *and* $\nu \in FPIIT(X)$. *Then* $\mu \cap \nu \in FPIIT(X)$.

Proof 3.35 *Let* $x, y, z \in X$. *Then*

$$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x).$$

Also

$$\begin{aligned}(\mu \cap \nu)(x * z) &= T(\mu(x * z), \nu(x * z)) \\ &\geq T(T(\mu(((x * z) * z) * (y * z)), \mu(y)), T(\nu(((x * z) * z) * (y * z)), \nu(y))) \\ &= T(T(\mu(((x * z) * z) * (y * z)), \nu(((x * z) * z) * (y * z))), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(((x * z) * z) * (y * z))), (\mu \cap \nu)(y))\end{aligned}$$

so

$$(\mu \cap \nu)(x * z) \geq T((\mu \cap \nu)(((x * z) * z) * (y * z))), (\mu \cap \nu)(y)).$$

Then $\mu \cap \nu \in FPIIT(X)$.

Proposition 3.36 *Let* $\mu \in FPIIT(X)$ *and* $\nu \in FPIIT(Y)$. *Then* $\mu \times \nu \in FPIIT(X \times Y)$.

Proof 3.37 *Let* $(x, y) \in X \times Y$. *Then*

$$(\mu \times \nu)(0, 0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(y)) = (\mu \times \nu)(x, y).$$

Also let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times Y$. *Then*

$$\begin{aligned}
&(\mu\times\nu)((x_1,x_2)*(z_1,z_2)) = (\mu\times\nu)(x_1*z_1,x_2*z_2) = T(\mu(x_1*z_1),\nu(x_2*z_2))\\
&\geq T(T(\mu(((x_1*z_1)*z_1)*(y_1*z_1)),\mu(y_1)),T(\nu(((x_2*z_2)*z_2)*(y_2*z_2)),\nu(y_2)))\\
&= T(T(\nu(((x_1*z_1)*z_1)*(y_1*z_1)),\nu(((x_2*z_2)*z_2)*(y_2*z_2))),T(\mu(y_1),\nu(y_2)))\\
&= T((\mu\times\nu)(((x_1*z_1)*z_1)*(y_1*z_1),((x_2*z_2)*z_2)*(y_2*z_2)),(\mu\times\nu)(y_1,y_2))\\
&= T((\mu\times\nu)((((x_1,x_2)*(z_1,z_2))*(z_1,z_2))*((y_1,y_2)*(z_1,z_2))),(\mu\times\nu)(y_1,y_2))
\end{aligned}$$

Therefore $\mu\times\nu\in FPIIT(X\times Y)$.

Proposition 3.38 *If* $\mu\in FPIIT(X)$ *and* $\varphi:X\to Y$ *be a homomorphism of* BCI*-algebras, then* $\varphi(\mu)\in FPIIT(Y)$.

Proof 3.39 *Let* $x\in X$ *and* $y\in Y$ *with* $\varphi(x)=y$. *Now*

$$\varphi(\mu)(0)=\sup\{\mu(0)\mid 0\in X,\varphi(0)=0\}\geq\sup\{\mu(x)\mid x\in X,\varphi(x)=y\}=\varphi(\mu)(y).$$

Also let $x_i\in X$ *such that* $\varphi(x_i)=y_i$ *and* $i=1,2,3$. *Then*

$$\begin{aligned}
&\varphi(\mu)(y_1*y_2)=\sup\{\mu(x_1*x_2)\mid x_i\in X,\varphi(x_i)=y_i\}\\
&\geq\sup\{T(\mu(((x_1*x_2)*x_2)*(x_3*x_2)),\mu(x_3))\mid x_i\in X,\varphi(x_i)=y_i\}\\
&=T(\sup\{\mu(((x_1*x_2)*x_2)*(x_3*x_2))\mid x_i\in X,\varphi(x_i)=y_i\},\sup\{\mu(x_3)\mid x_3\in X,\varphi(x_3)=y_3\})\\
&=T(\varphi(\mu)(((y_1*y_2)*y_2)*(y_3*y_2)),\varphi(\mu)(y_3))
\end{aligned}$$

therefore

$$\varphi(\mu)(y_1*y_2)\geq T(\varphi(\mu)(((y_1*y_2)*y_2)*(y_3*y_2)),\varphi(\mu)(y_3)).$$

Therefore $\varphi(\mu)\in FPIIT(Y)$.

Proposition 3.40 *If* $\nu\in FPIIT(Y)$ *and* $\varphi:X\to Y$ *be a homomorphism of* BCI*-algebras, then* $\varphi^{-1}(\nu)\in FPIIT(X)$.

Proof 3.41 *Let* $x\in X$. *Then*

$$\varphi^{-1}(\nu)(0)=\nu(\varphi(0))\geq\nu(\varphi(x))=\varphi^{-1}(\nu)(x).$$

Let $x_1,x_2,x_3\in X$. *As*

$$\begin{aligned}
\varphi^{-1}(\nu)(x_1*x_2)&=\nu(\varphi(x_1*x_2))\\
&=\nu(\varphi(x_1)*\varphi(x_2))\\
&\geq T(\nu(((\varphi(x_1)*\varphi(x_2))*\varphi(x_2))*(\varphi(x_3)*\varphi(x_2)))),\nu(\varphi(x_3)))\\
&=T(\nu(\varphi)(((x_1*x_2)*x_2)*(x_3*x_2)),\nu(\varphi(x_3)))\\
&=T(\varphi^{-1}(\nu)(((x_1*x_2)*x_2)*(x_3*x_2)),\varphi^{-1}(\nu)(x_3))
\end{aligned}$$

so

$$\varphi^{-1}(\nu)(x_1 * x_2) \geq T(\varphi^{-1}(\nu)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \varphi^{-1}(\nu)(x_3)).$$

Therefore $\varphi^{-1}(\nu) \in FPIIT(X)$.

Proposition 3.42 *If* $\mu \in FPIIT(X)$, *then* $\mu \in FIT(X)$.

Proof 3.43 *Let* $x, y, z \in X$ *and* $\mu \in FPIIT(X)$. *Then*
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x * z) \geq T(\mu(((x * z) * z) * (y * z)), \mu(y))$,
now in (2) let $z = 0$ *then* $\mu(x * 0) \geq T(\mu(((x * 0) * 0) * (y * 0)), \mu(y))$ *which means that* $\mu(x) \geq T(\mu(x * y), \mu(y))$. *Therefore* $\mu \in FIT(X)$.

Example 3.44 *Consider the* BCI*-algebra* $X = \{0, 1, 2, 3, 4\}$ *with the following caley table:*

$*$	*0*	*1*	*2*	*3*	*4*
0	*0*	*0*	*0*	*0*	*0*
1	*1*	*0*	*1*	*0*	*0*
2	*2*	*2*	*0*	*0*	*0*
3	*3*	*3*	*3*	*0*	*0*
4	*4*	*3*	*4*	*1*	*0*

Then $(X, *, 0)$ *is a* BCI*-algebra. Define* $\mu : X \to [0, 1]$ *as*

$$\mu(x) = \begin{cases} 1 & \textit{if } x = 0, 2 \\ t & \textit{if } x = 1, 3, 4 \end{cases}$$

such that $t \in (0, 1)$. *Let*

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \textit{if } x + y > 1 \\ 0 & \textit{otherwise.} \end{cases}$$

for all x,y $\in [0, 1]$ *then* $\mu \in FIT(X)$ *but* $\mu \notin FPIIT(X)$ *because: as we let* $x = 4, z = 3, y = 2$ *so from* $\mu(x * z) \geq T(\mu(((x * z) * z) * (y * z)), \mu(y))$ *we get that* $t \geq 1$ *and this is a contradition with* $t \in (0, 1)$.

Proposition 3.45 *Let* $\mu \in FIT(X)$. *Then* $\mu \in FST(X)$.

Proof 3.46 *We know that* $x * y \leq x$ *and from Proposition 3.20 we get that* $\mu(x * y) \geq \mu(x)$. *Now*

$$\mu(x * y) \geq \mu(x) \geq T(\mu(x * y), \mu(y)) \geq T(\mu(x), \mu(y))$$

and then $\mu \in FST(X)$.

Remark 3.47 *The converse of Proposition 3.45 may not be true. For example in Example 3.2 we have that* (X) *but since* $\mu(b) = 0.25T(\mu(b*a), \mu(a)) = T(\mu(a), \mu(a)) = \mu(a) = 0.55$ *so* $\mu \notin FIT(X)$.

Proposition 3.48 *Let* $\mu \in FST(X)$. *If* $\mu(x) \geq T(\mu(y), \mu(z))$ *and* $x * y \leq z$ *for all* $x, y, z \in X$, *then* $\mu \in FIT(X)$.

Proof 3.49 *As Proposition 3.5 we get that* $\mu(0) \geq \mu(x)$. *As* $x * (x * y) \leq y$ *so* $\mu(x) \geq T(\mu(x * y), \mu(y))$ *(From the hypothesis). Then* $\mu \in FIT(X)$.

4 Conclusion and open problem

In this paper, as using t-norms, we defined fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCI-algebras and we investigated fundamental properties of them. Now one can introduce fuzzy subalgebras, fuzzy ideals and fuzzy positive implicative ideals of BCC-algebras and obtain some results about them as we did for BCI-algebras and this can be an open problem.

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