

# A New Class of Triple Interval Numbers

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## Abstract

*In this study, we introduce the concept of triple interval valued sequence space  $\bar{l}^3(p)$ , where  $p=(p_{uvw})$  is a triple sequence of bounded strictly positive numbers and study its different properties like completeness, solidness, converge free etc. We prove some inclusion relations also.*

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## 1 Introduction

Interval arithmetic was first suggested by Dwyer [1] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [2] in 1959 and Moore and Yang [3] 1962. Furthermore, Moore and others [4], [5], [6] and [7] have developed applications to differential equations.

Chiao in [8] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [9] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. A set consisting of a closed interval of real numbers  $x$  such that  $a \leq x \leq b$  is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analytical properties. We denote the set of all real valued closed intervals by  $IR$ . Any elements of  $IR$  is a closed interval and denoted by  $\bar{A}$ . That is  $\bar{A} = \{x \in \mathbb{R}: a \leq x \leq b\}$ . An interval number  $\bar{A}$  is a closed subset of real numbers [8]. Let  $x_l$  and  $x_r$  be first and last points of  $\bar{A}$  interval number, respectively. For  $\bar{A}, \bar{B} \in IR$ , we have  $\bar{A} = \bar{B} \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$ .  $\bar{A} + \bar{B} = \{x \in \mathbb{R}: x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$ , and if  $\alpha \geq 0$ , then  $\alpha\bar{A} = \{x \in \mathbb{R}: \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$  and if  $\alpha < 0$ , then  $\alpha\bar{A} = \{x \in \mathbb{R}: \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$ ,

$$\begin{aligned} \bar{A} \cdot \bar{B} &= \{x \in \mathbb{R}: \min\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\} \leq x \\ &\leq \max\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\}\}. \end{aligned}$$

The set of all interval numbers  $IR$  is a complete metric space defined by

$$d(\bar{A}, \bar{B}) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} [10].$$

In the special case  $\bar{A} = [a, a]$  and  $\bar{B} = [b, b]$ , we obtain usual metric of  $R$ .

Let us define transformation  $f: \mathbb{N} \rightarrow R$  by  $k \rightarrow f(k) = \bar{A}_k$ ,  $\bar{A} = (\bar{A}_k)$ . Then  $\bar{A} = (\bar{A}_k)$  is called sequence of interval numbers. The  $\bar{A}_k$  is called  $k^{th}$  term of sequence  $\bar{A} = (\bar{A}_k)$ .  $w^i$  denotes the set of all interval numbers with real terms and the algebraic properties of  $w^i$  can be found in [11].

An interval valued sequence space  $\bar{E}$  is said to be solid if  $\bar{B} = (\bar{B}_k) \in \bar{E}$  whenever  $|\bar{B}_k| \leq |\bar{A}_k|$  for all  $k \in \mathbb{N}$  and  $\bar{A} = (\bar{A}_k) \in \bar{E}$ .

An interval valued sequence space  $\bar{E}$  is said to be monotone if  $\bar{E}$  contains the canonical pre-image of all its step spaces.

An interval valued sequence space  $\bar{E}$  is said to be convergence free if  $\bar{B} = (\bar{B}_k) \in \bar{E}$  whenever  $\bar{A} = (\bar{A}_k) \in \bar{E}$  and  $\bar{A}_k = \bar{0}$  implies  $\bar{B}_k = \bar{0}$ .

## 2 Main Results

Throughout the study,  $p = (p_{uvw})$  is a triple sequence of bounded strictly positive numbers. We define the following interval valued triple sequence space:

$$\bar{l}^3(p) = \left\{ \bar{A} = (\bar{A}_{uvw}) : \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} < \infty \right\}$$

and if  $p_{uvw} = 1$  for all  $u, v, w \in \mathbb{N}$ , then we have

$$\bar{l}^3 = \left\{ \bar{A} = (\bar{A}_{uvw}) : \sum_{u,v,w=1}^{\infty} d(\bar{A}_{uvw}, \bar{0}) < \infty \right\}.$$

**Lemma.** If a sequence space  $E$  is solid, then it is monotone.

**Theorem 1.** The sequence space  $\bar{l}^3(p)$  is a complete metric space with respect to the metric defined by

$$\rho(\bar{A}, \bar{B}) = \left[ \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{B}_{uvw})]^{p_{uvw}} \right]^{\frac{1}{M}},$$

where  $M = \max(1, \sup_{u,v,w} p_{uvw})$ .

**Proof.** Let  $(\bar{x}^i)$  be a Cauchy sequence in  $\bar{l}^3(p)$ . Then for a given  $\varepsilon > 0$ , there exists  $n_o \in \mathbb{N}$ , such that

$$\rho(\bar{A}_{uvw}^i, \bar{A}_{uvw}^j) < \varepsilon \text{ for all } i, j \geq n_o.$$

Then

$$\left[ \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}^i, \bar{A}_{uvw}^j)]^{p_{uvw}} \right]^{\frac{1}{M}} < \varepsilon \text{ for all } i, j \geq n_o \dots (2.1)$$

$$\Rightarrow \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}^i, \bar{A}_{uvw}^j)]^{p_{uvw}} < \varepsilon^M \text{ for all } i, j \geq n_o$$

$$\Rightarrow [d(\bar{A}_{uvw}^i, \bar{A}_{uvw}^j)] < \varepsilon \text{ for all } i, j \geq n_o \text{ and all } u, v, w \in \mathbb{N}.$$

This means that  $(\bar{A}_{uvw}^i)$  is a Cauchy sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is a Banach space,  $(\bar{A}_{uvw}^i)$  is convergent. Now, let

$$\lim \bar{A}_{uvw}^i = \bar{A}_{uvw} \text{ for each } u, v, w \in \mathbb{N} \text{ and } \bar{A} = (\bar{A}_{uvw}).$$

Taking limit as  $j \rightarrow \infty$  in (2.1), we have

$$\rho(\bar{A}^i, \bar{A}) < \varepsilon \text{ for all } i \geq n_o.$$

Now for all  $i \geq n_o$

$$\rho(\bar{A}, \bar{0}) \leq \rho(\bar{A}, \bar{A}^i) + \rho(\bar{A}^i, \bar{0}) < \infty.$$

Thus  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(p)$  and so  $\bar{l}^3(p)$  is complete. This completes the proof.

**Theorem 2.** The sequence space  $\bar{l}^3(p)$  is solid as well as monotone.

**Proof.** Let  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(p)$  and  $\bar{B} = (\bar{B}_{uvw}) \in \bar{l}^3(p)$  be a interval valued sequence such that  $|\bar{B}_{uvw}| \leq |\bar{A}_{uvw}|$  for all  $u, v, w \in \mathbb{N}$ . Then

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} < \infty$$

and

$$\sum_{u,v,w=1}^{\infty} [d(\bar{B}_{uvw}, \bar{0})]^{p_{uvw}} \leq \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} < \infty$$

Thus  $\bar{B} = (\bar{B}_{uvw}) \in \bar{l}^3(p)$  and so  $\bar{l}^3(p)$  is solid. Also by Lemma, it follows that the space  $\bar{l}^3(p)$  is monotone. This completes the proof.

**Theorem 3.** The sequence space  $\bar{l}^3(p)$  is not convergence free in general.

**Proof.** The result follows from the following example.

**Example.** Let  $p_{uvw}=2$  for all  $u, v, w \in \mathbb{N}$ . We consider the triple interval sequence  $\bar{A} = (\bar{A}_{uvw})$  as follows:

$$\bar{A}_{uvw} = \left[ \frac{-1}{(uvw)^2}, 0 \right], \text{ for all } k \in \mathbb{N}.$$

Then

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} \leq \sum_{u,v,w=1}^{\infty} \left( \frac{1}{(uvw)^2} \right)^2 < \infty.$$

So  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(p)$ . Now let us consider the interval sequence  $\bar{B} = (\bar{B}_{uvw})$  defined as follows:

$$\bar{B}_{uvw} = [-(uvw)^2, (uvw)^2], \text{ for all } u, v, w \in \mathbb{N}.$$

Then

$$\sum_{u,v,w=1}^{\infty} [d(\bar{B}_{uvw}, \bar{0})]^{p_{uvw}} \leq \sum_{u,v,w=1}^{\infty} ((uvw)^2)^2 = \infty,$$

so  $\bar{B} = (\bar{B}_{uvw}) \notin \bar{l}^3(p)$ . Hence  $\bar{l}^3(p)$  is not convergence free. This completes the proof.

**Theorem 4.**

(a) If  $0 < q_{uvw} \leq p_{uvw} < 1$  for all  $u, v, w \in \mathbb{N}$ , then  $\bar{l}^3(p) \subset \bar{l}^3(q)$  and the inclusion is proper.

(b) If  $0 < \inf_{u,v,w} p_{uvw} \leq p_{uvw}$  for all  $u, v, w \in \mathbb{N}$ , then  $\bar{l}^3(p) \subset \bar{l}^3$  and if  $1 < p_{uvw} \leq \sup_{u,v,w} p_{uvw} < \infty$  for all  $u, v, w \in \mathbb{N}$ , then  $\bar{l}^3 \subset \bar{l}^3(p)$ .

**Proof.(a)** Let  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(p)$ . Then

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} < \infty.$$

So there exists  $n_o \in \mathbb{N}$  such that

$$[d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} < 1, \text{ for all } u, v, w \geq n_o.$$

Thus

$$[d(\bar{A}_{uvw}, \bar{0})]^{q_{uvw}} < [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} \quad \text{for all } u, v, w \geq n_0$$

and so,

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{q_{uvw}} < \infty.$$

Thus  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(p)$ . The inclusion is strict and it follows from the following example.

**Example.** We consider the triple interval sequence  $\bar{A} = (\bar{A}_{uvw})$  as follows:

$$\bar{A}_{uvw} = \left[ \frac{-1}{\sqrt{3uvw}}, 0 \right], \quad \text{for all } u, v, w \in \mathbb{N}$$

and

$$q_{uvw} = 2 + \frac{1}{uvw} \quad \text{and} \quad p_{uvw} = 1, \quad \text{for all } u, v, w \in \mathbb{N}.$$

Then

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{q_{uvw}} = \sum_{u,v,w=1}^{\infty} \left( \frac{1}{\sqrt{3uvw}} \right)^{2 + \frac{1}{uvw}} < \infty,$$

and so  $\bar{A} = (\bar{A}_{uvw}) \in \bar{l}^3(q)$ , but

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} = \sum_{u,v,w=1}^{\infty} \frac{1}{\sqrt{3uvw}} = \infty,$$

so  $\bar{A} = (\bar{A}_{uvw}) \notin \bar{l}^3(p)$ . This completes the proof.

(b) The first part of the result follows from the inequality

$$\sum_{u,v,w=1}^{\infty} d(\bar{A}_{uvw}, \bar{0}) \leq \sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}}$$

and the second part of the result follows from the inequality

$$\sum_{u,v,w=1}^{\infty} [d(\bar{A}_{uvw}, \bar{0})]^{p_{uvw}} \leq \sum_{u,v,w=1}^{\infty} d(\bar{A}_{uvw}, \bar{0}).$$

## Open Problem

We introduced triple interval valued sequence space  $\bar{l}^3(p)$ , and studied its different properties like completeness, solidness, converge free etc. We also proved some inclusion relations about them. For the reference sections, consider the following introduction described the main results are motivating the research. Thus it will be possible to benefit from this article for new studies.

## References

- [1] P. S. Dwyer, *Linear Computation*, New York, Wiley, 1951.
- [2] R. E. Moore, *Automatic Error Analysis in Digital Computation*, LSMD-48421, Lockheed Missiles and Space Company, 1959.
- [3] R. E. Moore and C. T. Yang, *Interval Analysis I*, LMSD-285875, Lockheed Missiles and Space Company, 1962.
- [4] P. S. Dwyer, Errors of matrix computation, simultaneous equations and eigenvalues, *National Bureau of Standards Applied Mathematics Series* **29** (1953) 49–58.
- [5] A. Esi, Lacunary sequence spaces of interval numbers, *Thai J. Math.*, **10(2)** (2012) 445–451.
- [6] P. S. Fischer, Automatic propagated and round-off error analysis, *paper presented at the 13th National Meeting of the Association of Computing Machinery*, June 1958.
- [7] R. E. Moore and C. T. Yang, Theory of an interval algebra and its application to numeric analysis, *RAAG Memories II*, Gaukutsu Bunken Fukeyu-kai, Tokyo, 1958.
- [8] Kuo-Ping Chiao, Fundamental properties of interval vector max-norm, *Tamsui Oxf. J. Math. Sci.* **18(2)**(2002) 219–233.

- [9] M. Şengönül and A. Eryılmaz, On the sequence spaces of interval numbers, *Thai J. Math.*, **8(3)** (2010), 503-510.
- [10] A. R. Freedman, J. J. Sember and M. Raphael, Some Cesaro type summability, *Proc. London Math. Soc.* **37(3)** (1978) 508–520.
- [11] S. Markov, Quasilinear spaces and their relation to vector spaces, *Electronic Journal of Mathematics of Computation* **2(1)** (2005).