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On New Classes of Triple Sequence Spaces Defined By Orlicz Function

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Abstract

In this paper, we introduce triple sequence spaces via Orlicz function and examine some properties of the resulting these spaces.

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0.1 Introduction

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

Recall in [11] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called the modulus function and characterized by Ruckle [16] .An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct the sequence space

$l_M = \left\{ (x_i) : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) < \infty, for some r > 0 \right\}$,

which is a Banach space normed by

$\|(x_i)\| = inf \left\{ r > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) \leq 1 \right\}$.

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [5], Esi [1], Esi and Subramanian [2], Esi and Et [3], Parashar and Choudhary [12] and many others.

By the convergence of a triple sequence we mean the convergence on the Pringsheim sense [4] that is , a triple sequence $x = (x_{\text{ijl}})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{\text{ijl}} - L| < \varepsilon$ whenever $i, j, l > n$. We shall write more briefly as "$P - convergent$".

The concept of paranormed sequences was studied by Nakano [9] and Simons [13] at the initial stage. Later on it was studied by many others.

The triple sequence $x = (x_{\text{ijl}})$ is bounded if there exists a positive number M such that $|x_{\text{ijl}}| < M$ for all $i, j, l \in \mathbb{N}$. Let l_∞^3 the space of all bounded triple sequences such that

$\|x_{\text{ijl}}\|_{(\infty,3)} = \sup_{i,j,l} |x_{\text{ijl}}| < \infty$.

Throughout the paper, $w^3(X)$ denotes the spaces of all triple sequences in X, where (X, q) denotes a seminormed space, seminormed by q. The zero double sequence is denoted by θ in X.

2.**DEFINITIONSANDBACKGROUND**

Let P_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements and $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in$ N.

The sequence space $m(\phi)$ introduced by Sargent [15] is defined as follows:

$m(\phi) = \left\{x = (x_k) : \|x_k\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{i \in \sigma} |x_i| < \infty\right\}$

and studied some of its properties and obtained its relationship with the space l_p.

A triple sequence space E is said to be solid or normal if $(\alpha_{\text{ijl}} x_{\text{ijl}}) \in E$, whenever $(x_{\text{ijl}}) \in E$ for all triple sequences (α_{ijl}) of scalars such that $|\alpha_{\text{ijl}}| \leq 1$ for all $i, j, l \in$ N.

A triple sequence space E is said to be symmetric if $(x_{\text{ijl}}) \in E$ implies $\left(x_{\pi(i)\pi(j)\pi(l)}\right) \in E$, where π is a permutation of the elements of N.

Let $K = \{(i_n, j_k, l_m) : n, k, m \in \mathbb{N}; i_1 < i_2 < i_3 < ...., j_1 < j_2 < j_3 < ...and$

$l_1 < l_2 < l_3 < ... \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and E be a triple sequence space. A $K - step$ space of E is a sequence space

$\lambda_K^E = \{(x_{i_n j_k l_m}) : (x_{\text{ijl}}) \in E\}$.

A canonical pre-image of a sequence $(x_{\text{ijl}}) \in E$ is a sequence $(y_{\text{ijl}}) \in E$ defined as follows:

$$y_{\text{ijl}} = \begin{cases} x_{\text{ijl}}, if\ (i,j,l) \in K \\ 0, \quad \text{otherwise} \end{cases}.$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A triple sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.

Lemma. A double sequence space E is solid implies E is monotone.

Let P_{stq} denotes the class of all subsets of $\mathbb{N}\times \mathbb{N}\times\mathbb{N}$, those do not contain more that $s\times t\times q$ elements. Throughout the paper $\{\phi_{n,m,p}\}$ represent a non-decreasing triple sequence of real numbers such that $n\phi_{n+1,m,p}\leq (n+1)\,\phi_{n,m,p}$ and $m\phi_{n,m+1,p}\leq (m+1)\,\phi_{n,m,p}$ and $p\phi_{n,m,p+1}\leq (p+1)\,\phi_{n,m,p}$.

In this paper we introduce the following triple sequence spaces: Let M be an Orlicz function and a $p=(p_{\mathrm{ijl}})$ be a bounded triple sequence of positive real numbers such that $0<H_o=\inf_{i,j,l}p_{\mathrm{ijl}}\leq p_{\mathrm{ijl}}\leq \sup_{i,j,l}p_{\mathrm{ijl}}=H<\infty$, then

$l_{\infty}^{3}(M,q,p)=\left\{x=(x_{\mathrm{ijl}})\in w^{3}(X):\sup_{i,j,l}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}}<\infty, for\ some\ r>0\right\},$

$l_{p}^{3}(M,q)=\left\{x=(x_{\mathrm{ijl}})\in w^{3}(X):\sum_{i,j.l=1,1,1}^{\infty}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}}<\infty, for\ r>0\right\},$

$m^{3}(M,\phi,q,p)=\{x=(x_{\mathrm{ijl}})\in w^{3}(X):\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}}<\infty,$

for some $r>0\}$.

The following inequality will be used throughout the paper

$|a_{\mathrm{ijl}}+b_{\mathrm{ijl}}|^{p_{\mathrm{ijl}}}\leq\max\left(1,2^{H-1}\right)\left(|a_{\mathrm{ijl}}|^{p_{\mathrm{ijl}}}+|b_{\mathrm{ijl}}|^{p_{\mathrm{ijl}}}\right)$

where a_{ijl} and b_{ijl} are complex numbers and $H=\sup_{i,j.l}p_{\mathrm{ijl}}<\infty$.

It is easy to see that the triple sequence space $l_{p}^{3}(M,q)$ is a seminormed space, seminormed by

$g((x_{\mathrm{ijl}}))=inf\left\{r^{\frac{p_{\mathrm{ijl}}}{J}}>0:\sum_{i,j,l=1,1,1}^{\infty}M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\leq 1\right\}, whereJ=\max\left(1,2^{H-1}\right)$

3.MAIN RESULTS

In this section we prove some results involving the triple sequence spaces $m^{3}(M,\phi,q,p)\,,l_{p}^{3}(M,q)$ and $l_{\infty}^{3}(M,q,p)$.

Theorem 3.1. $m^{3}(M,\phi,q,p)$ and $l_{\infty}^{3}(M,q,p)$ are linear spaces.

Proof. Let $(x_{\mathrm{ijl}}),(y_{\mathrm{ijl}})\in m^{3}(M,\phi,q,p)$ and $\alpha,\beta\in$ C. Then there exists positive numbers r_1 and r_2 such that

$\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r_1}\right)\right)\right]^{p_{\mathrm{ijl}}}<\infty$

and

$\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{y_{\mathrm{ijl}}}{r_2}\right)\right)\right]^{p_{\mathrm{ijl}}}<\infty.$

Let $r_3=max\,(2\,|\alpha|\,r_1,2\,|\beta|\,r_2)$. Since M is non-decreasing convex function and q is a seminorm, we have

$\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{\alpha x_{\mathrm{ijl}}+\beta y_{\mathrm{ijl}}}{r_3}\right)\right)\right]^{p_{\mathrm{ijl}}}$

$\leq\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{\alpha x_{\mathrm{ijl}}}{r_3}\right)+q\left(\frac{\beta y_{\mathrm{ijl}}}{r_3}\right)\right)\right]^{p_{\mathrm{ijl}}}$

$\leq max\left(1,2^{H-1}\right)\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r_1}\right)\right)\right]^{p_{\mathrm{ij}}}+max\left(1,2^{H-1}\right)\left[M\left(q\left(\frac{y_{\mathrm{ijl}}}{r_2}\right)\right)\right]^{p_{\mathrm{ijl}}}.$

So,

$\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{\alpha x_{\mathrm{ijl}}+\beta y_{\mathrm{ijl}}}{r_3}\right)\right)\right]^{p_{\mathrm{ijl}}}$

$\leq\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r_1}\right)\right)\right]^{p_{\mathrm{ijl}}}+$

$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{y_{\text{ijl}}}{r_2}\right)\right)\right]^{p_{\text{ijl}}}$
$< \infty$,
therefore $(\alpha x_{\text{ijl}} + \beta y_{\text{ijl}}) \in m^3(M,\phi,q,p)$. Hence $m^3(M,\phi,q,p)$ is a linear space.

The proof for the case $l^3_\infty(M,q,p)$ is a routine work in view of the above proof.

Theorem 3.2. The space $m^3(M,\phi,q,p)$ is a seminormed space, seminormed by
$h((x_{\text{ijl}})) = inf\left\{r^{\frac{p_{\text{ijl}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right) \leq 1\right\}, where J = \max\left(1, 2^{H-1}\right).$

Proof. Clearly $h((x_{\text{ijl}})) \geq 0$ for all $(x_{\text{ijl}}) \in m^2(M,\phi,q,p)$ and $h(\theta) = 0$. Let $r_1 > 0$ and $r_2 > 0$ be such that
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{r_1}\right)\right) \leq 1$
and
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{y_{\text{ijl}}}{r_2}\right)\right) \leq 1.$
Let $r = r_1 + r_2$. Then we have
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}+y_{\text{ijl}}}{r}\right)\right)$
$= \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}+y_{\text{ijl}}}{r_1+r_2}\right)\right)$
$\leq \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left\{\frac{r_1}{r_1+r_2} M\left(q\left(\frac{x_{\text{ijl}}}{r_1}\right)\right) + \frac{r_2}{r_1+r_2} M\left(q\left(\frac{y_{\text{ijl}}}{r_2}\right)\right)\right\}$
$\leq \left(\frac{r_1}{r_1+r_2}\right) \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{r_1}\right)\right)$
$+ \left(\frac{r_2}{r_1+r_2}\right) \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{y_{\text{ijl}}}{r_2}\right)\right)$
$\leq 1.$
Since the $r's$ are nonnegative, so we have
$h((x_{\text{ijl}}) + (y_{\text{ijl}})) = inf\left\{r^{\frac{p_{\text{ijl}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}+y_{\text{ijl}}}{r}\right)\right) \leq 1\right\}$
$\leq \inf\left\{r_1^{\frac{p_{\text{ijl}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{r_1}\right)\right) \leq 1\right\}$
$+ \inf\left\{r_2^{\frac{p_{\text{ijl}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{r_2}\right)\right) \leq 1\right\}$
$= h((x_{\text{ijl}})) + h((y_{\text{ijl}}))$.
Next for $\lambda \in \mathbb{C}$, without loss of generality, let $\lambda \neq 0$, then
$h((\lambda x_{\text{ijl}})) = inf\left\{r^{\frac{p_{\text{ijl}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{\lambda x_{\text{ijl}}}{r}\right)\right) \leq 1\right\}$
$= inf\left\{(|\lambda|\rho)^{\frac{p_{\text{ilj}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} M\left(q\left(\frac{x_{\text{ijl}}}{\rho}\right)\right) \leq 1\right\}, where \rho = \frac{r}{|\lambda|}$
Hence we get
$h((\lambda x_{\text{ijl}})) \leq \max(1, |\lambda|)\, h((x_{\text{ijl}}))$.
This completes the proof of the theorem.
The proof of the following result is a consequence of the above theorem.

Proposition 3.3. The double sequence space $l_\infty^3(M,q,p)$ is a seminormed space, seminormed by

$f((x_{ijl})) = inf\left\{r^{\frac{p_{ijl}}{J}} > 0 : \sup_{i,j,l} M\left(q\left(\frac{x_{ijl}}{r}\right)\right) \leq 1\right\}, whereJ = \max\left(1, 2^{H-1}\right)$

Theorem 3.4. $m^3(M,\phi,q,p) \subset m^3(M,\psi,q,p)$ if and only if $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$.

Proof. Let $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$ and $(x_{ijl}) \in m^3(M,\phi,q,p)$. Then

$\sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, forsomer > 0.$

So,

$\sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\psi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}}$

$\leq \left(\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}}\right) \sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty.$

Therefore $(x_{ijl}) \in m^3(M,\psi,q,p)$. Hence $m^3(M,\phi,q,p) \subset m^3(M,\psi,q,p)$.

Conversely, let $m^3(M,\phi,q,p) \subset m^3(M,\psi,q,p)$. Suppose that $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} = \infty$. Then there exists a sequence of natural numbers (s_{ijl}) such that $P - \lim_{i,j,l} \frac{\phi_{s_i t_j q_l}}{\psi_{s_i t_j q_l}} = \infty$. Let $(x_{ijl}) \in m^3(M,\phi,q,p)$. Then there exists $r > 0$ such that

$\sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,s)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty.$

Now, we have

$\sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\psi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}}$

$\geq \left(\sup_{i,j,l} \frac{\phi_{s_i,t_j,q_l}}{\psi_{s_i,t_j,q_l}}\right) \sup_{i,j,y\geq 1,\sigma\in P_{s_i t_j s_y}} \frac{1}{\phi_{s_i,t_j,,q_l}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ijl}} = \infty.$

Therefore $(x_{ijl}) \notin m^3(M,\psi,q,p)$. This is a contradiction. Hence $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$.

The following result is a consequence of Theorem 3.4.

Corollary 3.5. Let M be an Orlicz function. Then $m^3(M,\phi,q,p) = m^3(M,\psi,q,p)$ if and only if $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$ and $\sup_{s,t.,q} \frac{\psi_{s,t,q}}{\phi_{s,t,q}} < \infty$ for all $s,t,q = 1,2,....$

Theorem 3.6. Let M, M_1, M_2 be Orlicz functions satisfying $\Delta_2 - condition$. Then

(i) $m^3(M_1,\phi,q,p) \subset m^3(\text{Mo}M_1,\phi,q,p)$,

(ii) $m^3(M_1,\phi,q,p) \cap m^3(M_2,\phi,q,p) \subset m^3(M_1+M_2,\phi,q,p)$.

Proof. **(i)** Let $(x_{ijl}) \in m^3(M_1,\phi,q,p)$. Then there exists $r > 0$ such that

$\sup_{s,t,q\geq 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty.$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t < \delta$. Let $y_{ij} = M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)$ and for any $\sigma \in P_{stq}$, let

$\sum_{(i,j,l)\in\sigma} [M(y_{ijl})]^{p_{ijl}} = \sum_1 [M(y_{ijl})]^{p_{ijl}} + \sum_2 [M(y_{ijl})]^{p_{ijl}}$

where the first summation is over $y_{ijl} \leq \delta$ and the second is over $y_{ijl} > \delta$. By the remark we have

$\sum_1 M(y_{ijl}) \leq \max\left(1, [M(1)]^H\right) \sum_1 (y_{ijl})^{p_{ilj}} \leq \max\left(1, [M(2)]^H\right) \sum_1 (y_{ijl})^{p_{ijl}}$

(3.1)

For $y_{\text{ij}} > \delta$
$y_{\text{ijl}} < y_{\text{ijl}}\delta^{-1} \leq 1 + y_{\text{ijl}}\delta^{-1}$,
since M is non-decreasing and convex, so
$M(y_{\text{ijl}}) < M(1 + y_{\text{ijl}}\delta^{-1}) < \frac{1}{2}M(2) + \frac{1}{2}M(2y_{\text{ijl}}\delta^{-1})$.
Since M satisfies $\Delta_2 - condition$, so
$M(y_{\text{ijl}}) < \frac{K}{2}y_{\text{ijl}}\delta^{-1}M(2) + \frac{K}{2}y_{\text{ijl}}\delta^{-1}M(2) = Ky_{\text{ijl}}\delta^{-1}M(2)$.
Hence,
$\sum_2 [M(y_{\text{ijl}})]^{p_{\text{ijl}}} \leq \max\left(1, [K\delta^{-1}M(2)]^H\right)\sum_2 (y_{\text{ijl}})^{p_{\text{ijl}}}$. (3.2)

By (3.1) and (3.2) we have $(x_{\text{ijl}}) \in m^3(\text{Mo}M_1, \phi, q, p)$. Thus $m^3(M_1, \phi, q, p) \subset m^3(\text{Mo}M_1, \phi, q, p)$.

(ii) Let $(x_{\text{ijl}}) \in m^3(M_1, \phi, q, p) \cap m^3(M_2, \phi, q, p)$. Then there exists $r > 0$ such that
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}} < \infty$
and
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_2\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ilj}}} < \infty$.
The rest of the proof follows from the equality
$\sum_{(i,j,l)\in\sigma} \left[(M_1 + M_2)\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}}$
$\leq \max\left(1, 2^{H-1}\right)\sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}} + \max\left(1, 2^{H-1}\right)\sum_{(i,j,l)\in\sigma} \left[M_2\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}}$.
This completes the proof.

Taking $M_1(x) = x$ in above theorem, we have the following result.

Corollary 3.7. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^3(\phi, q, p) \subset m^3(M, \phi, q, p)$.

From Theorem 3.4. and Corollary 3.7., we have:

Corollary 3.8. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^3(\phi, q, p) \subset m^3(M, \psi, q, p)$ if and only if $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$.

Theorem 3.9. The double space $m^3(M, \phi, q, p)$ is solid and symmetric.

Proof. Let $(x_{\text{ijl}}) \in m^3(M, \phi, q, p)$. Then
$\sup_{s,t,q\geq 1,\sigma\in P_{\text{stq}}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}} < \infty$. (3.3)

Let (λ_{ij}) be a double sequence of scalars with $|\lambda_{\text{ijl}}| \leq 1$ for all $i, j, l \in$ N. Then the result follows from (3.3) and the following inequality
$\sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{\lambda_{\text{ijl}}x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}} \leq \sum_{(i,j,l)\in\sigma} \left[|\lambda_{\text{ijl}}| M\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}}$ $(bytheRemark)$
$\leq \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{\text{ijl}}}{r}\right)\right)\right]^{p_{\text{ijl}}}$.

The symmetricity of the space follows from the definition of the triple space $m^3(M, \phi, q, p)$ and symmetric triple sequence space.

The following result follows from Theorem 3.9 and the Lemma.

Corollary 3.10. The triple space $m^3(M, \phi, q, p)$ is monotone.

The proof of the following result is a routine work.

Proposition 3.11. The triple spaces $l_p^3(M, q)$ and $l_\infty^3(M, q, p)$ are solid and as such are monotone.

Theorem 3.12. $l_p^3(M, q) \subset m^3(M, \phi, q, p) \subset l_\infty^3(M, q, p)$.

Proof. Let $(x_{\mathrm{ijl}}) \in l_p^3(M,q)$. Then we have
$\sum_{i,j,l=1,1,1}^{\infty}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}} < \infty, for\ some\ r > 0.$ (3.4)
Since $\{\phi_{n,m}\}$ is monotonic increasing, so we have
$\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}} \leq \frac{1}{\phi_{1,1,1}}\sum_{(i,j,)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ij}}l} < \infty.$
Hence
$\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ij}}l} < \infty.$
Thus $(x_{\mathrm{ijl}}) \in m^3(M,\phi,q,p)$. Therefore $l_p^3(M,q) \subset m^3(M,\phi,q,p)$. Next let $(x_{\mathrm{ijl}}) \in m^3(M,\phi,q,p)$. Then we have
$\sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}} < \infty, for\ some\ r > 0.$
So,
$\sup_{i,j,l\in\mathbb{N}}\frac{1}{\phi_{1,1,1}}\left[M\left(q\left(\frac{x_{\mathrm{ijl}}}{r}\right)\right)\right]^{p_{\mathrm{ijl}}} < \infty, forsomer > 0\,(\text{on taking cardinality of}\sigma tobe1).$
Therefore $(x_{\mathrm{ijl}}) \in l_\infty^3(M,q,p)$. Hence $m^3(M,\phi,q,p) \subset l_\infty^3(M,q,p)$. This completes the proof.

Theorem 3.13.(i) $m^3(M,\phi,q,p) = l_p^3(M,q)$ if and only if $\sup_{s,t,q\geq 1}\phi_{s,t,q} < \infty$.

(ii) $m^3(M,\phi,q,p) = l_\infty^3(M,q,p)$ if and only if $\sup_{s,t,q\geq 1}\frac{\mathrm{stq}}{\phi_{s,t,q}} < \infty$.

Proof.(i) It is clear that $m^3(M,\psi,q,p) = l_p^3(M,q)$ when $\psi_{s,t,q} = 1$ for all $s,t,q \in$ N. By Theorem 3.4., $m^3(M,\phi,q,p) \subset m^3(M,\psi,q,p)$ if and only if $\sup_{s,t}\frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$ i.e. $\sup_{s,t,q}\phi_{s,t,q} < \infty$. By Theorem 3.11., $m^3(M,\phi,q,p)$ $= l_p^3(M,q)$ if and only if $\sup_{s,t,q}\phi_{s,t,q} < \infty$.

(ii) We have $m^3(M,\psi,q,p) = l_\infty^3(M,q)$ if $\psi_{s,t,q} = \mathrm{stq}$ for all $s,t,q \in$ N. By Theorem 3.4. and Theorem 3.11., it follows that $m^3(M,\phi,q,p) = l_\infty^3(M,q,p)$ if and only if $\sup_{s,t,q\geq 1}\frac{\mathrm{stq}}{\phi_{s,t,q}} < \infty$.

This completes the proof.

The proof of the following result is routine work.

Proposition 3.14. Let M be an Orlicz function, q_1 and q_2 be seminorms. Then

(i) $m^3(M,\phi,q_1,p) \cap m^3(M,\phi,q_2,p) \subset m^2(M,\phi,q_1+q_2,p)$,

(ii) If q_1 is stronger than q_2, then $m^3(M,\phi,q_1,p) \subset m^3(M,\phi,q_2,p)$,

(iii) $l_\infty^3(M,q_1,p) \cap l_\infty^3(M,q_2,p) \subset l_\infty^3(M,q_1+q_2,p)$,

(iv) If q_1 is stronger than q_2, then $l_\infty^3(M,q_1,p) \subset l_\infty^3(M,q_2,p)$,

(v) $l_p^3(M,q_1) \cap l_p^3(M,q_2) \subset l_p^3(M,q_1+q_2)$,

(vi) If q_1 is stronger than q_2, then $l_p^3(M,q_1) \subset l_p^3(M,q_2)$.

4.PARTICULARCASES

If one considers a normed linear space $(X,\|.\|)$ instead of a seminormed space (X,q), then one will get $m^3(M,\phi,\|.\|,p)$, which will be a normed linear space, normed by

$\|(x_{\mathrm{ij}})\| = inf\left\{r^{\frac{p_{\mathrm{ij}}}{J}} > 0 : \sup_{s,t,q\geq 1,\sigma\in P_{\mathrm{stq}}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}M\left(\left\|\frac{x_{\mathrm{ij}}}{r}\right\|\right) \leq 1\right\}, whereJ = \max\left(1,2^{H-1}\right).$

The triple space $m^3(M, \phi, \|.\|, p)$ will be a solid, monotone and symmetric space. Further most of the results proved in the previous section will be true for this space too.

Open Problem: In the introduced spaces can Orlicz functions be replaced by other functions ? Whether the proof of seminormed spaces can be checked for other spaces ?

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