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The Study of a New Subclass of Harmonic Functions Associated with Mittag-Leffler-type Poisson Distribution Series

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Abstract

In this paper a subclass of p-valent harmonic functions associated with Mittag-Leffler-type Poisson Distribution Series in the open unit disc has been introduced and some properties as coefficients estimate,extreme points, distortion bounds and closure theorems have been studied.

Keywords: *Open Unit disk, Multivalent Functions, Harmonic Functions, Mittag-Leffler-type Poisson Distribution Series.*

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1 Introduction

A continuous function F = H + iG is a complex valued harmonic function in a complex domain C , if both U and V are real harmonic in C. In any simply connected domain $D \subseteq C$,we can write F=H+$\bar{G}$,. We call H the analytic part and G the co-analytic part of F. A necessary and sufficient condition for F to be locally univalent and sense-preserving in D is that $|H'(z)| > |G'(z)|$ in D (see Clunie and Sheil-Small [3]).
Denote by C(p) the class of functions F=H+$\bar{G}$, that are harmonic multivalent and sense-preserving in the unit disk D=$\{z \in C : |z| < 1\}$. The class C(p) was studied by Ahuja and Jahangiri [1] and class C(p) for p=1 was defined and studied by Jahangiri et. al. in [6]. For F=H+$\bar{G} \in C(p)$, we may express the

analytic functions H and G as:

$$H(z) = z^p + \sum_{n=p+1}^{\infty} |a_n| z^n, G(z) = \sum_{n=p}^{\infty} |b_n| z^n (p \in N) \tag{1}$$

where $z \in D, |b_p| < 1$.
Now Mittag-Leffler function $E_\alpha(z)$ introduced by Mittag-Leffler [7] is given as $E_\alpha(z) = \sum_{n=p}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, (z \in D, Re(\alpha) > 0)$.
Wiman[8] has introduced a general function $E_{\alpha,\beta}$ as $E_{\alpha,\beta} = \sum_{n=p}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, (z, \alpha, \beta \in D, Re(\beta) > 0, Re(\alpha) > 0)$.
The normalization of the Mittag-Leffler function [4] is given as $\mathbb{E}_{\alpha,\beta}(z) = \Gamma(\beta) z E_{\alpha,\beta} = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n$
where z,$\alpha, \beta \in D : \beta \neq 0, -1, -2, ...$ and $Re(\beta) > 0, Re(\alpha) > 0$.
The probability mass function of the Mittag-Leffler-type Poisson Distribution is given as
$P(X = r) = \frac{m^r}{\Gamma(\alpha k + \beta) \mathbb{E}_{\alpha,\beta}(m)}, r = 0, 1, 2, 3....$
where $m > 0, \alpha > 0$ and $\beta > 0$.
Using this normalized form of the Mittag-Leffler-type Poisson Distribution, Frasin et.al.[5] has defined an operator on class C(p) for p=1 as
$I^m_{\alpha,\beta} F(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}(m)} c_n z^n; z \in D$
Motivated with this work we defined a new subclass $\mathbb{R}^{m,p}_{\alpha,\beta}$ of Harmonic functions $F(z) \in C(p)$, which satisfy the conditions

$$Re\left\{ \frac{(1 + \delta e^{i\Phi}) z (I^m_{\alpha,\beta} F(z))'}{p(I^m_{\alpha,\beta} F(z))} - \delta e^{i\Phi} \right\} \geq \gamma, \tag{2}$$

for ($\alpha \geq 0$ and $0 \leq \beta < 1, 0 \leq \gamma < 1$)
or equivalently

$$Re\left\{ \frac{(1 + \delta e^{i\Phi}) z (I^m_{\alpha,\beta} F(z))' - p\delta e^{i\Phi} (I^m_{\alpha,\beta} F(z))}{p(I^m_{\alpha,\beta} F(z))} \right\} \geq \gamma \tag{3}$$

where

$$I^m_{\alpha,\beta} F(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}(m)} c_n z^n, z \in D \tag{4}$$

We also let $C\mathbb{R}^{m,p}_{\alpha,\beta} = \mathbb{R}^{m,p}_{\alpha,\beta} \bigcap C(p)$.
In this paper we determine the Coefficients inequality, Distortion theorem, Extreme points and Clousre proerty for the class $C\mathbb{R}^{m,p}_{\alpha,\beta}$.

2 COEFFICIENTS INEQUALITY

We will first derive a sufficient condition for the functions of the class $\mathbb{R}^{m,p}_{\alpha,\beta}$.
Theorem 2.1: Let F=H+$\bar{G}$ (H and G being given by 1.1). If

$$\sum_{n=p+1}^{\infty}\left[\left\{\frac{n(1+\delta)-p(\gamma+\delta)}{p(1-\gamma)}\right\}|a_n|+\left\{\frac{n(1+\delta)+p(\gamma+\delta)}{p(1-\gamma)}\right\}|b_n|\right]\Gamma^{m,n}_{\alpha,\beta} \quad (5)$$

$\leq 1-\left\{\frac{1+\gamma+2\delta}{1-\gamma}\right\}\Gamma^{m,p}_{\alpha,\beta}|b_p|$,
then $F\in\mathbb{R}^{m,p}_{\alpha,\beta}$, where

$$\Gamma^{m,n}_{\alpha,\beta}=\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)},. \quad (6)$$

Proof: Let F=H+$\bar{G}$ in (2.1), we get

$$\text{Re}\left\{\frac{(1+\delta e^{i\Phi})\left[z(I^m_{\alpha,\beta}H(z))'-\overline{z(I^m_{\alpha,\beta}G(z))'}\right]-p\delta e^{i\Phi}(I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})}{p((I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})}\right\}=$$

$$Re\left\{\frac{A(z)}{B(z)}\right\}$$

where A(z)= $(1+\delta e^{i\Phi})\left[z(I^m_{\alpha,\beta}H(z))'-\overline{z(I^m_{\alpha,\beta}G(z))'}\right]-p\delta e^{i\Phi}(I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})$
and B(z)=$p((I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})$
Using the fact that Re $\omega(z)\geq\gamma$ if and only if $|1-\gamma+\omega|\geq|1+\gamma-\omega|$ [2], it is sufficient to show that
$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\geq 0$.
Substituting for A(z) and B(z), we get

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|=|(1+\delta e^{i\Phi})\left[z(I^m_{\alpha,\beta}H(z))'-\overline{z(I^m_{\alpha,\beta}G(z))'}\right]-$$
$$p\delta e^{i\Phi}(I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})+(1-\gamma)\left[p((I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})\right]|-$$
$$|(1+\delta e^{i\Phi})\left[z(I^m_{\alpha,\beta}H(z))'-\overline{z(I^m_{\alpha,\beta}G(z))'}\right]-p\delta e^{i\Phi}(I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})-(1+$$
$$\gamma)\left[p((I^m_{\alpha,\beta}H(z)+\overline{I^m_{\alpha,\beta}G(z)})\right]|$$
$$=|(1+\delta e^{i\Phi})\left[pz^p+\sum_{n=p+1}^{\infty}n\Gamma^{m,n}_{\alpha,\beta}a_nz^{n-1}-\sum_{n=p}^{\infty}n\Gamma^{m,n}_{\alpha,\beta}\overline{b_nz^{n-1}}\right]$$
$$-\delta e^{i\Phi}\left[pz^p+\sum_{n=p+1}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}a_nz^n+\sum_{n=p}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}\overline{b_nz^n}\right]$$
$$+(1-\gamma)\left[pz^p+\sum_{n=p+1}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}a_nz^n+\sum_{n=p}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}\overline{b_nz^n}\right]|-$$
$$|(1+\delta e^{i\Phi})\left[pz^p+\sum_{n=p+1}^{\infty}n\Gamma^{m,n}_{\alpha,\beta}a_nz^{n-1}-\sum_{n=p}^{\infty}n\Gamma^{m,n}_{\alpha,\beta}\overline{b_nz^{n-1}}\right]$$
$$-\delta e^{i\Phi}\left[pz^p+\sum_{n=p+1}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}a_nz^n+\sum_{n=p}^{\infty}p\Gamma^{m,n}_{\alpha,\beta}\overline{b_nz^n}\right]$$

$- (1+\gamma)\left[pz^p + \sum_{n=p+1}^{\infty} p\Gamma_{\alpha,\beta}^{m,n} a_n z^n + \sum_{n=p}^{\infty} p\Gamma_{\alpha,\beta}^{m,n} \overline{b_n z^n}\right] |$
$= | \left[2p(1-\gamma)e^{i\Phi}\right] z^p + \sum_{n=p+1}^{\infty} 2\left[n(1+\delta) - p(\gamma+\delta)e^{i\Phi}\right] \Gamma_{\alpha,\beta}^{m,n} a_n z^n$
$- \sum_{n=p}^{\infty} 2\left[n(1+\delta) + p(\gamma+\delta)e^{i\Phi}\right] \Gamma_{\alpha,\beta}^{m,n} \overline{b_n z^n}|$
$\geq [2p(1-\gamma)]\, |z|^p - \sum_{n=p+1}^{\infty} 2\left[n(1+\delta) - p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |a_n||z|^n$
$- \sum_{n=p}^{\infty} 2\left[n(1+\delta) + p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |b_n||z|^n$
$= 2p(1-\gamma)|z|^p - \sum_{n=p+1}^{\infty} 2\left[n(1+\delta) - p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |a_n||z|^n$
$- \sum_{n=p}^{\infty} 2\left[n(1+\delta) + p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |b_n||z|^n$
$\geq 2p[(1-\gamma) - (1+\gamma+2\delta)\Gamma_{\alpha,\beta}^{m,p}|b_p|]|z|^p$
$- \sum_{n=p+1}^{\infty} 2\left[n(1+\delta) - p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |a_n||z|^n$
$- \sum_{n=p+1}^{\infty} 2\left[n(1+\delta) + p(\gamma+\delta)\right] \Gamma_{\alpha,\beta}^{m,n} |b_n||z|^n \geq 0.$
By virtue of (2.1), this implies that $F \in \mathbb{R}_{\alpha,\beta}^{m,p}$.
Theorem 2.2: Let F=H+$\bar{G} \in C(p)$. Then $F \in C\mathbb{R}_{\alpha,\beta}^{m,p}$ if and only if

$$\sum_{n=p+1}^{\infty}\left[\left\{\frac{n(1+\delta)-p(\gamma+\delta)}{p(1-\gamma)}\right\}|a_n| + \left\{\frac{n(1+\delta)+p(\gamma+\delta)}{p(1-\gamma)}\right\}|b_n|\right]\Gamma_{\alpha,\beta}^{m,n} \quad (7)$$

$\leq 1 - \left\{\dfrac{1+\gamma+2\delta}{1-\gamma}\right\}\Gamma_{\alpha,\beta}^{m,p}|b_p|$ where $0 \leq \gamma < 1$.
Proof: Since $C\mathbb{R}_{\alpha,\beta}^{m,p} \subset \mathbb{R}_{\alpha,\beta}^{m,p}$, we only need to prove 'only if' part of the theorem. To this end, for functions f of the form (1.1) with condition (1.3), we notice the condition

$$\mathrm{Re}\left\{\frac{(1+\delta e^{i\Phi})z(I_{\alpha,\beta}^{m}F(z))'}{p(I_{\alpha,\beta}^{m}F(z))} - (\delta e^{i\Phi}+\gamma)\right\} \geq 0.$$

The above inequality is equivalent to
$Re[(1+\delta e^{i\Phi})\left[pz^p + \sum_{n=p+1}^{\infty} n\Gamma_{\alpha,\beta}^{m,n}(a_n z^n - \sum_{n=p}^{\infty} n\Gamma_{\alpha,\beta}^{m,n}\overline{b_n z^n}\right]$
$-(\delta e^{i\Phi}+\gamma)\left[pz^p + \sum_{n=p+1}^{\infty} p\Gamma_{\alpha,\beta}^{m,n} a_n z^n + \sum_{n=p}^{\infty} p\Gamma_{\alpha,\beta}^{m,n}\overline{b_n z^n}\right]$
$\times\left[pz^p + \sum_{n=p+1}^{\infty} p\Gamma_{\alpha,\beta}^{m,n} a_n z^n + \sum_{n=p}^{\infty} p\Gamma_{\alpha,\beta}^{m,n}\overline{b_n z^n}\right]^{-1}] \geq 0.$
$Re[p(1-\gamma) + \sum_{n=p+1}^{\infty}\left\{(n-p\gamma) + \delta(n-p)e^{i\Phi}\right\}\Gamma_{\alpha,\beta}^{m,n} a_n z^{n-p}$
-$\frac{1}{z^p}\sum_{n=p}^{\infty}\left\{(n+p\gamma) + \delta(n+p)e^{i\Phi}\right\}\Gamma_{\alpha,\beta}^{m,n}\overline{b_n}\overline{z}^n]$
$\times[p + \sum_{n=p+1}^{\infty} p\Gamma_{\alpha,\beta}^{m,n} a_n z^{n-p} + \frac{1}{z^p}\sum_{n=p}^{\infty} p\Gamma_{\alpha,\beta}^{m,n}\overline{b_n z^n}]^{-1} \geq 0.$
Thus condition must hold for all values of z, such that $|z| = r < 1$. Choosing Φ according to condition and noting that $Re(-\delta e^{i\Phi}) \geq -\delta|e^{i\Phi}| = -\delta$, the above inequality reduces to

$$|p(1-\gamma) - p\left\{1+\gamma+2\delta\right\}\Gamma_{\alpha,\beta}^{m,p}|b_p| \quad (8)$$

$- \sum_{n=p+1}^{\infty}\left[\left\{n(1+\delta) - p(\gamma+\delta)\right\}|a_n| + \left\{n(1+\delta) + p(\gamma+\delta)\right\}|b_n|\right]\Gamma_{\alpha,\beta}^{m,n} r^{n-p}|$

$\times[p+\sum_{n=p+1}^{\infty} p\Gamma_{\alpha,\beta}^{m,n}|a_n|r^{n-p}+\sum_{n=p}^{\infty} p\Gamma_{\alpha,\beta}^{m,n}|b_n|r^{n-p}]]^{-1}\geq 0.$
Letting $r \longrightarrow 1^-$ and if the condition (2.3) does not hold, then the numerator in (2.4) is negative. This contradicts our assumptions that $F \in C\mathbb{R}_{\alpha,\beta}^{m,p}$. Hence $F \in C\mathbb{R}_{\alpha,\beta}^{m,p}$.
For $\sum_{n=p+1}^{\infty}|A_n|+\sum_{n=p}^{\infty}|B_n|=1$, the harmonic univalent function
$F(z)=z^p+\sum_{n=p+1}^{\infty}\dfrac{(1-\gamma)}{\{n(1+\delta)-p(\gamma+\delta)\}\,\Gamma_{\alpha,\beta}^{m,n}}A_n z^n$
$+\sum_{n=p}^{\infty}\dfrac{(1-\gamma)}{\{n(1+\delta)+p(\gamma+\delta)\}\,\Gamma_{\alpha,\beta}^{m,n}}\overline{B_n z^n}$
shows that equality in the coefficient bound given by (2.3) is sharp.

3 DISTORTION THEOREM

Theorem 3.1: Let the function F(z)=H(z)+$\bar{G(z)}$ defined by (1.1) be in the class $C\mathbb{R}_{\alpha,\beta}^{m,p}$. Then for $|z|=r<1$, we have

$$|F(z)|\leq(1+|b_p|)r^p+\frac{p(1-\gamma)}{\Gamma_{\alpha,\beta}^{m,p+1}}\left[1-\left\{\frac{1+\gamma+2\delta}{1-\gamma}\right\}\Gamma_{\alpha,\beta}^{m,p}|b_p|\right]r^{p+1} \qquad (9)$$

and

$$|F(z)|\geq(1-|b_p|)r^p-\frac{p(1-\gamma)}{\Gamma_{\alpha,\beta}^{m,p+1}}\left[1-\left\{\frac{1+\gamma+2\delta}{1-\gamma}\right\}\Gamma_{\alpha,\beta}^{m,p}|b_p|\right]r^{p+1}. \qquad (10)$$

The result is sharp.
Proof: We prove only the left hand inequality. Let $F \in C\mathbb{R}_{\alpha,\beta}^{m,p}$. Taking the absolute value of F(z), we have
$|F(z)|\geq(1-|b_p|)r^p-\sum_{n=p+1}^{\infty}\{|a_n|+|b_n|\}\,r^n$
$\geq(1-|b_p|)r^p-r^{p+1}\sum_{n=p+1}^{\infty}\{|a_n|+|b_n|\}$
$\geq(1-|b_p|)r^p-\dfrac{p(1-\gamma)}{\Gamma_{\alpha,\beta}^{m,p+1}}r^{p+1}\sum_{n=p+1}^{\infty}\dfrac{\Gamma_{\alpha,\beta}^{m,n}}{p(1-\gamma)}\{|a_n|+|b_n|\}$
$\geq(1-|b_p|)r^p-\dfrac{p(1-\gamma)}{\Gamma_{\alpha,\beta}^{m,p+1}}\sum_{n=p+1}^{\infty}\left[\dfrac{n(1+\delta)-p(\gamma+\delta)}{p(1-\gamma)}|a_n|+\dfrac{n(1+\delta)+p(\gamma+\delta)}{p(1-\gamma)}|b_n|\right]\Gamma_{\alpha,\beta}^{m,n}r^{p+1}$
$\geq(1-|b_p|)r^p-\dfrac{p(1-\gamma)}{\Gamma_{\alpha,\beta}^{m,p+1}}\left[1-\left\{\dfrac{1+\gamma+2\delta}{1-\gamma}\right\}\Gamma_{\alpha,\beta}^{m,p}|b_p|\right]r^{p+1}.$

4 EXTREME POINTS

Theorem 4.1: A function $F = H + \bar{G} \in C\mathbb{R}^{m,p}_{\alpha,\beta}$ if and only if F(z) can be expressed in the form

$$F(z) = \sum_{n=p}^{\infty} (X_n H_n + Y_n G_n) \tag{11}$$

where

$$H_p(z) = z^p, H_n(z) = z^p + \frac{(1-\gamma)}{\{n(1+\delta) - p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} z^n (n \geq p+1) \tag{12}$$

$G_n(z) = z^p + \dfrac{(1-\gamma)}{\{n(1+\delta) + p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} z^n (n \geq p),$

$\Gamma^{m,n}_{\alpha,\beta} F(z)$ is given by (1.4) and $\sum_{n=p}^{\infty} (X_n + Y_n) = 1$ with $X_n \geq 0, Y_n \geq 0$.

In particular the extreme points of $C\mathbb{R}^{m,p}_{\alpha,\beta}$ are H_n and G_n.

Proof: Let F(z) be of the form (4.1). Then we have

$F(z) = \sum_{n=p}^{\infty} (X_n + Y_n) z^p + \sum_{n=p+1}^{\infty} \dfrac{(1-\gamma)}{\{n(1+\delta) - p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} X_n z^n$

$+ \sum_{n=p}^{\infty} \dfrac{(1-\gamma)}{\{n(1+\delta) + p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} Y_n z^n$

$= z^p + \sum_{n=p+1}^{\infty} \dfrac{(1-\gamma)}{\{n(1+\delta) - p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} X_n z^n$

$+ \sum_{n=p}^{\infty} \dfrac{(1-\gamma)}{\{n(1+\delta) + p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} Y_n z^n$

Furthermore, let

$|a_n| = \dfrac{(1-\gamma)}{\{n(1+\delta) - p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} X_n$

and $|b_n| = \dfrac{(1-\gamma)}{\{n(1+\delta) + p(\gamma+\delta)\}\,\Gamma^{m,n}_{\alpha,\beta}} Y_n$.

Now, $\sum_{n=p+1}^{\infty} \left\{ \dfrac{n(1+\delta) - p(\gamma+\delta)}{(1-\gamma)} \right\} \Gamma^{m,n}_{\alpha,\beta} |a_n| +$

$\sum_{n=p}^{\infty} \left\{ \dfrac{n(1+\delta) + p(\gamma+\delta)}{(1-\gamma)} \right\} \Gamma^{m,n}_{\alpha,\beta} |b_n|$

$= \sum_{n=p+1}^{\infty} X_n + \sum_{n=p}^{\infty} Y_n = 1 - X_p \leq 1$. So $F \in C\mathbb{R}^{m,p}_{\alpha,\beta}$.

Conversely, suppose that $F \in C\mathbb{R}^{m,p}_{\alpha,\beta}$.

Setting $X_n = \left\{ \dfrac{n(1+\delta) - p(\gamma+\delta)}{(1-\gamma)} \right\} \Gamma^{m,n}_{\alpha,\beta} |a_n| \ (n \geq p+1)$

$Y_n = \left\{ \dfrac{n(1+\delta) + p(\gamma+\delta)}{(1-\gamma)} \right\} \Gamma^{m,n}_{\alpha,\beta} |b_n| \ (n \geq p)$

and $X_p = 1 - \sum_{n=p+1}^{\infty} X_n - \sum_{n=p}^{\infty} Y_n$.

Therefore,
$F(z) = z^p + \sum_{n=p+1}^{\infty} |a_n| z^n + \sum_{n=p}^{\infty} |b_n| z^n$
$= z^p + \sum_{n=p+1}^{\infty} \frac{(1-\gamma)}{\{n(1+\delta) - p(\gamma+\delta)\} \Gamma_{\alpha,\beta}^{m,n}} X_n z^n$
$+ \sum_{n=p}^{\infty} \frac{(1-\gamma)}{\{n(1+\delta) + p(\gamma+\delta)\} \Gamma_{\alpha,\beta}^{m,n}} Y_n z^n$
$= z^p + \sum_{n=p+1}^{\infty} [X_n \{H_n(z) - z^p\}] + \sum_{n=p}^{\infty} [B_n \{G_n(z) - z^p\}]$
So $F(z) = \sum_{n=p}^{\infty} (X_n H_n(z) + Y_n G_n(z))$.

5 Open Problem

The open problem here is to find more normalized form of Mittag-Leffler-type Poisson Distribution series and to find more classes of Harmonic functions satisfying some conditions.Also to find some more properties like Radius of starlikness, Radiud of convexity etc.

References

[1] Ahuja,O.P. and Jahangiri,J.M.; Multivalent harmonic starlike functions, Ann. Univ. Marie-Curie Sklodowska sect. A, 55(1)(2001).

[2] Aqlan, E.S.; Some problems connected with geometric function theory, Ph.D. Thesis(2004), Pune University, Pune.

[3] Clunie,J.G. and Shell-Small,T.;Harmonic univalent functions,Aun.Acad.Aci. Fenn. Ser. A.I. Math,9(1984), 3-25.

[4] Bansal D. and Prajapati J.K.;Certain geometric properties of the Mittag-Leffler functions, Complex Var. Elliptic Equ. 61(3),338-313,2002.

[5] Frasin B.A., Porwal S. and Yousef F; Subclasses of starlike and convex functions associated with Mittag-Leffler-type Poisson Distribution series, Montes Taurus J. Pure Appl. Math. 3(3), 147-154, 2021.

[6] Jahangiri,J.M., Murugusundaramoorthy,G. and Vijaya,K.; Salagean-type harmonic univalent functions, South. J. Pure Appl. Math., 2(2002), 77-82.

[7] Mittag-Leffler G.M.; Sur la nouvelle function E(x), C.R. Acad. Sci. Paris, 137, 554-558,1903.

[8] Wiman A.; Uber die Nullstellun der Funncktionen E(x), Acta Math. 29, 217-234, 1905.