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$QIFSN(G)$ and Strongest Relations, Cosets and Middle Cosets

Rasul Rasuliand Amir Shomali

Department of Mathematics,
Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran.
e-mail: Rasuli@pnu.ac.ir

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Abstract

In this paper a subclass of p-valent harmonic functions associated with Mittag-Leffler-type Poisson Distribution Series in the open unit disc has been introduced and some properties as coefficients estimate,extreme points, distortion bounds and closure theorems have been studied.

Keywords: *Open Unit disk, Multivalent Functions, Harmonic Functions, Mittag-Leffler-type Poisson Distribution Series.*
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1 Introduction

After the introduction of fuzzy sets by Zadeh [25], various notions of higher-order fuzzy sets have been proposed. Among them, intuitionistic fuzzy sets, introduced by Atanassov [3, 4], have drawn the attention of many researchers in the last decades. This is mainly due to the fact that intuitionistic fuzzy sets are consistent with human behavior, by reecting and modeling the hesitancy present in real-life situations. In fact, the fuzzy sets give the degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of nonmembership, and furthermore the sum of these two degrees is not greater than 1. Yuan and Lee [24] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Solairaju and

Nagarajan [23] introduced the notion of Q- fuzzy groups. Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on t-norm. In previous works [6-22], by using norms, the first author investigated some properties of fuzzy algebraic structures, specially, we defined and investigated Q-fuzzy subgroups, anti Q-fuzzy subgroups, Q-intuitionistic fuzzy subgroups with respect to norms [6, 7,8, 9, 20]. In this study, we We consider the concepts of strongest relations, cosets and middle cosets of Q-intuitionistic fuzzy subgroups with respect to norms(T and C) and investigate some of their properties. Moreover, we investigate some related properties of them under homomorphism and anti homomorphism.

2 preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer to [1, 2, 3, 4, 5, 6, 7, 8, 10, 20].

A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that
(1) if a and b belong to G then ab is also in G (closure),
(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
(3) there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),
(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Let G be a group. Let H be a non-empty subset of G. The following are equivalent:
(1) H is a subgroup of G.
(2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $[0, 1]^G$.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, IFS) in X such that

$\mu_A, \nu_A \in [0,1]^X$ and for all $x \in X$ we have $(\mu_A(x)+\nu_A(x)) \in [0,1]$. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0,1)$ and $U_X(x) = (1,0)$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.

Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then
(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) Equality:$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element)
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)
(T3) $T(x,y) = T(y,x)$ (commutativity)
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),
for all $x,y,z \in [0,1]$.

Let T be a t-norm. Then for all $x \in [0,1]$
(1) $T(x,0) = 0$.
(2) $T(0,0) = 0$.

(1) Standard intersection t-norm

$$T_m(x,y) = \min\{x,y\}.$$

(2) Bounded sum t-norm

$$T_b(x,y) = \max\{0, x+y-1\}.$$

(3) algebraic product t-norm

$$T_p(x,y) = xy.$$

(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm:

$$T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$$

for all $x, y \in [0,1]$.

Let T be a t-norm. Then

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(C1) $C(x,0) = x$
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$
(C3) $C(x,y) = C(y,x)$
(C4) $C(x, C(y,z)) = C(C(x,y), z)$,
for all $x, y, z \in [0,1]$.

Let C be a C-conorm. Then for all $x \in [0,1]$
(1) $C(x,1) = 1$.
(2) $C(0,0) = 0$.

(1) Standard union t-conorm

$$C_m(x,y) = \max\{x,y\}.$$

(2) Bounded sum t-conorm

$$C_b(x,y) = \min\{1, x+y\}.$$

(3) Algebraic sum t-conorm

$$C_p(x,y) = x + y - xy.$$

(4) Drastic t-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic t-norm.

(5) Nilpotent maximum t-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity)

$$C_{H_2}(x,y) = \frac{x+y}{1+xy}$$

is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm:

$$C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$$

for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-norm $T($ t-conorm $C)$ is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

Let C be a t-conorm. Then

$$C(C(x,y), C(w,z)) = C(C(x,w), C(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Let $(G, .)$ be a group and Q be a non empty set. An intuitionistic fuzzy set $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ is said to be a Q-intuitionistic fuzzy subgroup of G with respect to norms (t-norm T and t-conorm C) if the following conditions are satisfied:

(1)

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) \supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))),$$

(2)

$$A(x^{-1},q) = (\mu_A(x^{-1},q), \nu_A(x^{-1},q)) \supseteq A(x,q) = (\mu_A(x,q), \nu_A(x,q))$$

which mean:

(a) $\mu_A(xy,q) \geq T(\mu_A(x,q), \mu_A(y,q))$,

(b) $\nu_A(xy,q) \leq C(\nu_A(x,q), \nu_A(y,q))$,

(c) $\mu_A(x^{-1},q) \geq \mu_A(x,q)$,

(d) $\nu_A(x^{-1},q) \leq \nu_A(x,q)$,

for all $x, y \in G$ and $q \in Q$. Throughout this paper the set of all Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C) will be denoted by $QIFSN(G)$.

Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ such that T and C be idempotent. Then

$$A(e_G, q) \supseteq A(x,q)$$

for all $x \in G$ and $q \in Q$.

3 strongest relations, cosets and middle cosets of $QIFSN(G)$

Let $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ and $B = (\mu_B, \nu_B) \in IFS((G \times G) \times Q)$. We say that B is the strongest relation of G with respect to A if $B((x, y), q) = (\mu_B((x, y), q), \nu_B((x, y), q))$ such that $\mu_B((x, y), q) = T(\mu_A(x, q), \mu_A(y, q))$ and $\nu_B((x, y), q) = C(\nu_A(x, q), \nu_A(y, q))$ for all $(x, y) \in G \times G$ and $q \in Q$.

Let $A = (\mu_A, \nu_A) \in IFS(G\times Q)$ and $B = (\mu_B, \nu_B) \in IFS((G\times G)\times Q)$ such that B be the strongest relation of G with respect to A. Then $A \in QIFSN(G)$ if and only if $B \in QIFSN(G \times G)$. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and $(x_1, x_2), (y_1, y_2) \in G \times G$ and $q \in Q$.
(1)

$$\begin{aligned}
\mu_B((x_1, x_2)(y_1, y_2), q) &= \mu_B((x_1y_1, x_2y_2), q) \\
&= T(\mu_A(x_1y_1, q), \mu_A(x_2y_2, q)) \\
&\geq T(T(\mu_A(x_1, q), \mu_A(x_2, q)), T(\mu_A(y_1, q), \mu_A(y_2, q))) \\
&= T(T(\mu_A(x_1, q), \mu_A(y_1, q)), T(\mu_A(x_2, q), \mu_A(y_2, q))) \\
&= T(\mu_B((x_1, y_1), q), \mu_B((x_2, y_2), q))
\end{aligned}$$

and then

$$\mu_B((x_1, x_2)(y_1, y_2), q) \geq T(\mu_B((x_1, y_1), q), \mu_B((x_2, y_2), q)).$$

(2)

$$\begin{aligned}
\nu_B((x_1, x_2)(y_1, y_2), q) &= \nu_B((x_1y_1, x_2y_2), q) \\
&= C(\nu_A(x_1y_1, q), \nu_A(x_2y_2, q)) \\
&\leq C(C(\nu_A(x_1, q), \nu_A(x_2, q)), C(\nu_A(y_1, q), \nu_A(y_2, q))) \\
&= C(C(\nu_A(x_1, q), \nu_A(y_1, q)), C(\nu_A(x_2, q), \nu_A(y_2, q))) \\
&= C(\nu_B((x_1, y_1), q), \nu_B((x_2, y_2), q))
\end{aligned}$$

thus

$$\nu_B((x_1, x_2)(y_1, y_2), q) \leq C(\nu_B((x_1, y_1), q), \nu_B((x_2, y_2), q)).$$

Let $(x, y) \in G \times G$ and $q \in Q$.
(3)

$$\begin{aligned}
\mu_B((x, y)^{-1}, q) &= \mu_B((x^{-1}, y^{-1}), q) \\
&= T(\mu_A(x^{-1}, q), \mu_A(y^{-1}, q)) \\
&\geq T(\mu_A(x, q), \mu_A(y, q)) \\
&= \mu_B((x, y), q)
\end{aligned}$$

and so

$$\mu_B((x,y)^{-1},q) \geq \mu_B((x,y),q).$$

(4)

$$\begin{aligned}\nu_B((x,y)^{-1},q) &= \nu_B((x^{-1},y^{-1}),q)\\ &= C(\nu_A(x^{-1},q),\nu_A(y^{-1},q))\\ &\leq C(\nu_A(x,q),\nu_A(y,q))\\ &= \nu_B((x,y),q)\end{aligned}$$

and so

$$\nu_B((x,y)^{-1},q) \leq \nu_B((x,y),q).$$

Then from (1)-(4) we get thaat $B = (\mu_B, \nu_B) \in QIFSN(G \times G)$.
Conversely, let $B = (\mu_B, \nu_B) \in QIFSN(G \times G)$.
(1) Let $x_1, x_2, y_1, y_2 \in G$ with $x_2 = y_2 = e_G$ and $q \in Q$ then Proposition 2.17 give us that $A(e,q) \supseteq A(x_1y_1,q)$. Then
(1)

$$\begin{aligned}\mu_A(x_1y_1,q) &= T(\mu_A(x_1y_1,q),\mu_A(e_G,q))\\ &= T(\mu_A(x_1y_1,q),\mu_A(x_2y_2,q))\\ &= \mu_B((x_1y_1,x_2y_2),q)\\ &= \mu_B((x_1,x_2)(y_1,y_2),q)\\ &\geq T(\mu_B((x_1,x_2),q),\mu_B((y_1,y_2),q))\\ &= T(T(\mu_A(x_1,q),\mu_A(x_2,q)),T(\mu_A(y_1,q),\mu_A(y_2,q)))\\ &= T(T(\mu_A(x_1,q),\mu_A(e,q)),T(\mu_A(y_1,q),\mu_A(e,q)))\\ &\geq T(T(\mu_A(x_1,q),\mu_A(x_1,q)),T(\mu_A(y_1,q),\mu_A(y_1,q)))\\ &= T(\mu_A(x_1,q),\mu_A(y_1,q)))\end{aligned}$$

and thus

$$\mu_A(x_1y_1,q) \geq T(\mu_A(x_1,q),\mu_A(y_1,q))).$$

Also

$$\begin{aligned}
\nu_A(x_1y_1,q) &= C(\nu_A(x_1y_1,q),\nu_A(e_G,q))\\
&= C(\nu_A(x_1y_1,q),\nu_A(x_2y_2,q))\\
&= \nu_B((x_1y_1,x_2y_2),q)\\
&= \nu_B((x_1,x_2)(y_1,y_2),q)\\
&\le C(\nu_B((x_1,x_2),q),\nu_B((y_1,y_2),q))\\
&= C(C(\nu_A(x_1,q),\nu_A(x_2,q)),C(\nu_A(y_1,q),\nu_A(y_2,q)))\\
&= C(C(\nu_A(x_1,q),\nu_A(e,q)),C(\nu_A(y_1,q),\nu_A(e,q)))\\
&\le C(C(\nu_A(x_1,q),\nu_A(x_1,q)),C(\nu_A(y_1,q),\nu_A(y_1,q)))\\
&= C(\nu_A(x_1,q),\nu_A(y_1,q)))
\end{aligned}$$

thus

$$\nu_A(x_1y_1,q) \le C(\nu_A(x_1,q),\nu_A(y_1,q))).$$

(2) Let $x, y \in G$ with $y = e_G$ and $q \in Q$. Then as Proposition 2.17 we obtain

$$\begin{aligned}
\mu_A(x^{-1},q) &= T(\mu_A(x^{-1},q),\mu_A(e_G,q))\\
&\ge T(\mu_A(x^{-1},q),\mu_A(y^{-1},q))\\
&= \mu_B((x^{-1},y^{-1}),q)\\
&= \mu_B((x,y)^{-1},q)\\
&\ge \mu_B((x,y),q)\\
&= T(\mu_A(x,q),\mu_A(y,q))\\
&= T(\mu_A(x,q),\mu_A(e,q))\\
&\ge T(\mu_A(x,q),\mu_A(x,q))\\
&= \mu_A(x,q)
\end{aligned}$$

and then

$$\mu_A(x^{-1},q) \ge \mu_A(x,q).$$

Also

$$\begin{aligned}\nu_A(x^{-1},q) &= C(\nu_A(x^{-1},q),\nu_A(e_G,q))\\ &\leq C(\nu_A(x^{-1},q),\nu_A(y^{-1},q))\\ &= \nu_B((x^{-1},y^{-1}),q)\\ &= \nu_B((x,y)^{-1},q)\\ &\leq \nu_B((x,y),q)\\ &= C(\nu_A(x,q),\nu_A(y,q))\\ &= C(\nu_A(x,q),\nu_A(e,q))\\ &\leq C(\nu_A(x,q),\nu_A(x,q))\\ &= \nu_A(x,q)\end{aligned}$$

thus

$$\nu_A(x^{-1},q) \leq \nu_A(x,q).$$

Therefore from (1)-(2) we get that $A=(\mu_A,\nu_A)\in QIFSN(G)$.

Let $A=(\mu_A,\nu_A)\in QIFSN(G)$ then middle coset $aAb : G\times Q\to[0,1]$ is defined by

$$(aAb)(x,q)=(a\mu_A b,a\nu_A b)(x,q)=(\mu_A(a^{-1}xb^{-1},q),\nu_A(a^{-1}xb^{-1},q))$$

for all $x\in G, q\in Q$ and $a,b\in G$.

Let $A=(\mu_A,\nu_A)\in QIFSN(G)$. Then $aAa^{-1}\in QIFSN(G)$ for any $a\in G$. Let $a,x,y\in G$ and $q\in Q$. Then

$$\begin{aligned}(a\mu_A a^{-1})(xy,q) &= \mu_A(a^{-1}xya,q)\\ &= \mu_A(a^{-1}xaa^{-1}ya,q)\\ &\geq T(\mu_A(a^{-1}xa,q),\mu_A(a^{-1}ya,q))\\ &= T((a\mu_A a^{-1})(x,q),(a\mu_A a^{-1})(y,q))\end{aligned}$$

then

$$(a\mu_A a^{-1})(xy,q)\geq T((a\mu_A a^{-1})(x,q),(a\mu_A a^{-1})(y,q)).$$

Also

$$\begin{aligned}(a\nu_A a^{-1})(xy,q) &= \nu_A(a^{-1}xya,q)\\ &= \nu_A(a^{-1}xaa^{-1}ya,q)\\ &\leq C(\nu_A(a^{-1}xa,q),\nu_A(a^{-1}ya,q))\\ &= C((a\nu_A a^{-1})(x,q),(a\nu_A a^{-1})(y,q))\end{aligned}$$

then

$$(a\nu_A a^{-1})(xy,q)\leq C((a\nu_A a^{-1})(x,q),(a\nu_A a^{-1})(y,q)).$$

And

$$\begin{aligned}(a\mu_A a^{-1})(x^{-1},q) &= \mu_A(a^{-1}x^{-1}a,q)\\ &= \mu_A((a^{-1}xa)^{-1},q)\\ &\geq \mu_A(a^{-1}xa,q)\\ &= (a\mu_A a^{-1})(x,q)\end{aligned}$$

thus

$$(a\mu_A a^{-1})(x^{-1},q) \geq (a\mu_A a^{-1})(x,q).$$

Moreover

$$\begin{aligned}(a\nu_A a^{-1})(x^{-1},q) &= \nu_A(a^{-1}x^{-1}a,q)\\ &= \nu_A((a^{-1}xa)^{-1},q)\\ &\leq \nu_A(a^{-1}xa,q)\\ &= (a\nu_A a^{-1})(x,q)\end{aligned}$$

thus

$$(a\nu_A a^{-1})(x^{-1},q) \leq (a\nu_A a^{-1})(x,q).$$

Then $aAa^{-1} \in QIFSN(G)$ for any $a \in G$.

Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ then coset $aA = (a\mu_A, a\nu_A) : G \times Q \to [0,1]$ is defined by

$$(aA)(x,q) = (a\mu_A, a\nu_A)(x,q) = (\mu_A(a^{-1}x,q), \nu_A(a^{-1}x,q)) = A(a^{-1}x,q)$$

for all $x \in G, q \in Q$ and $a \in G$.

Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and T, C be idempotent norms. Then

$$xA = yA$$

if and only if

$$A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$$

for all $x, y \in G$ and $q \in Q$. Let $x, y, g \in G$.
If $xA = yA$, then $xA(x,q) = yA(x,q)$ then $A(x^{-1}x,q) = A(y^{-1}x,q)$ and so $A(e_G,q) = A(y^{-1}x,q)$.
Also as $xA = yA$ so $xA(y,q) = yA(y,q)$ and then $A(x^{-1}y,q) = A(y^{-1}y,q)$ then $A(x^{-1}y,q) = A(e_G,q)$. Therefore $A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$.
Conversely, let $A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$. Then

$$\begin{aligned}
x\mu_A(g,q) &= \mu_A(x^{-1}g,q)\\
&= \mu_A(x^{-1}yy^{-1}g,q)\\
&\geq T(\mu_A(x^{-1}y,q),\mu_A(y^{-1}g,q))\\
&= T(\mu_A(e_G,q),\mu_A(y^{-1}g,q))\\
&\geq T(\mu_A(y^{-1}g,q),\mu_A(y^{-1}g,q))\\
&= \mu_A(y^{-1}g,q)\\
&= y\mu_A(g,q)\\
&= \mu_A(y^{-1}g,q)\\
&= \mu_A(y^{-1}xx^{-1}g,q)\\
&\geq T(\mu_A(y^{-1}x,q),\mu_A(x^{-1}g,q))\\
&= T(\mu_A(e_G,q),\mu_A(x^{-1}g,q))\\
&\geq T(\mu_A(x^{-1}g,q),\mu_A(x^{-1}g,q))\\
&= \mu_A(x^{-1}g,q)\\
&= x\mu_A(g,q)
\end{aligned}$$

and then

$$x\mu_A(g,q) = y\mu_A(g,q).$$

Also

$$\begin{aligned}
x\nu_A(g,q) &= \nu_A(x^{-1}g,q)\\
&= \nu_A(x^{-1}yy^{-1}g,q)\\
&\leq C(\nu_A(x^{-1}y,q),\nu_A(y^{-1}g,q))\\
&= C(\nu_A(e_G,q),\nu_A(y^{-1}g,q))\\
&\leq C(\nu_A(y^{-1}g,q),\nu_A(y^{-1}g,q))\\
&= \nu_A(y^{-1}g,q)\\
&= y\nu_A(g,q)\\
&= \nu_A(y^{-1}g,q)\\
&= \nu_A(y^{-1}xx^{-1}g,q)\\
&\leq C(\nu_A(y^{-1}x,q),\nu_A(x^{-1}g,q))\\
&= C(\nu_A(e_G,q),\nu_A(x^{-1}g,q))\\
&\leq C(\nu_A(x^{-1}g,q),\nu_A(x^{-1}g,q))\\
&= \nu_A(x^{-1}g,q)\\
&= x\nu_A(g,q)
\end{aligned}$$

and then

$$x\nu_A(g,q) = y\nu_A(g,q).$$

Therefore $xA(g,q) = (x\mu_A(g,q), x\nu_A(g,q)) = (y\mu_A(g,q), y\nu_A(g,q)) = yA(g,q)$ then $xA = yA$.

Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and T, C be idempotent norms. If $xA = yA$, then $A(x,q) = A(y,q)$ for all $x, y \in G$ and $q \in Q$. As $xA = yA$, Proposition 3.6 gives us that $A(x^{-1}y, q) = A(y^{-1}x, q) = A(e_G, q)$ for all $x, y \in G$ and $q \in Q$. From

$$\begin{aligned}\mu_A(x,q) &= \mu_A(yy^{-1}x,q)\\ &\geq T(\mu_A(y,q), \mu_A(y^{-1}x,q))\\ &= T(\mu_A(y,q), \mu_A(e_G,q))\\ &\geq T(\mu_A(y,q), \mu_A(y,q))\\ &= \mu_A(y,q)\\ &= \mu_A(xx^{-1}y,q)\\ &\geq T(\mu_A(x,q), \mu_A(x^{-1}y,q))\\ &= T(\mu_A(x,q), \mu_A(e_G,q))\\ &\geq T(\mu_A(x,q), \mu_A(x,q))\\ &= \mu_A(x,q)\end{aligned}$$

we get that $\mu_A(x,q) = \mu_A(y,q)$. Also

$$\begin{aligned}\nu_A(x,q) &= \nu_A(yy^{-1}x,q)\\ &\leq C(\nu_A(y,q), \nu_A(y^{-1}x,q))\\ &= C(\nu_A(y,q), \nu_A(e_G,q))\\ &\leq C(\nu_A(y,q), \nu_A(y,q))\\ &= \nu_A(y,q)\\ &= \nu_A(xx^{-1}y,q)\\ &\leq C(\nu_A(x,q), \nu_A(x^{-1}y,q))\\ &= C(\nu_A(x,q), \nu_A(e_G,q))\\ &\leq C(\nu_A(x,q), \nu_A(x,q))\\ &= \nu_A(x,q)\end{aligned}$$

then $\nu_A(x,q) = \nu_A(y,q)$. Thus $A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = (\mu_A(y,q), \nu_A(y,q)) = A(y,q)$.

We say that $A = (\mu_A, \nu_A) \in QIFSN(G)$ is a normal if $\mu_A(xyx^{-1}, q) = \mu_A(y,q)$ and $\nu_A(xyx^{-1}, q) = \nu_A(y,q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NQIFSN(G)$ the set of all normal Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C).

If $A = (\mu_A, \nu_A) \in NQIFSN(G)$, then the set $\frac{G}{A} = \{xA : x \in G\}$ is a group with the operation $(xA)(yA) = (xy)A$. (1) If $x, y \in G$, then $xy \in G$. If $xA, yA \in \frac{G}{A}$ then

$$(xA)(yA) = (xy)A \in \frac{G}{A}.$$

(2) Let $x, y, z \in G$ then $x(yz) = (xy)z$. Now let $xA, yA, zA \in \frac{G}{A}$ so

$$\begin{aligned}(xA)[(yA)(zA)] &= [(xA)](yzA)\\ &= (xyz)A\\ &= (xy)zA\\ &= (xyA)(zA)\\ &= [(xyA)](zA)\\ &= [(xA)(yA)](zA).\end{aligned}$$

(3) Let $x \in G$ then $xe_G = e_Gx = x$. Thus

$$(xA)(e_GA) = (xe_GA) = (e_GxA) = xA.$$

If $x \in G$, then there is an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e_G$. If $xA \in \frac{G}{A}$, then there is an element $(xA)^{-1} = x^{-1}A \in \frac{G}{A}$ such that

$$(xA)(xA)^{-1} = (xA)(x^{-1}A) = (xx^{-1}A) = (x^{-1}xA) = e_GA = A.$$

Hence $\frac{G}{A}$ is a group.

Let $f : G \to H$ be a homomorphism of groups and let $B = (\mu_B, \nu_B) \in NQIFSN(H)$ and $A = (\mu_A, \nu_A) \in NQIFSN(G)$ be homomorphic pre-image of B. Then $\varphi : \frac{G}{A} \to \frac{H}{B}$ such that $\varphi(xA) = f(x)B$, for every $x \in G$, is an isomorphism of groups. Firstly, we prove that φ is a group homomorphism. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((xA)(yA)) = \varphi((xy)A) = f(xy)B = f(x)f(y)B = f(x)Bf(y)B = \varphi(xA)\varphi(yA)$$

and so φ is a group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(xA) = \varphi(yA)$, then $f(x)B = f(y)B$ and from Proposition 3.6 we get that

$$B(f(x)^{-1}f(y), q) = B(f(y)^{-1}f(x), q) = B(f(e_G), q)$$

and so

$$B(f(x^{-1})f(y), q) = B(f(y^{-1})f(x), q) = B(f(e_G), q)$$

and then

$$B(f(x^{-1}y), q) = B(f(y^{-1}x), q) = B(f(e_G), q)$$

which implies that

$$A(x^{-1}y, q) = A(y^{-1}x, q) = A(e_G, q)$$

and thus Proposition 3.6 gives us that $xA = yA$ which implies that φ is one-one. Therefore φ will be an isomorphism of groups.

Let $f : G \to H$ be an anti homomorphism of groups and let $B = (\mu_B, \nu_B) \in NQIFSN(H)$ and $A = (\mu_A, \nu_A) \in NQIFSN(G)$ be anti homomorphic pre-image of B. Then $\varphi : \frac{G}{A} \to \frac{H}{B}$ such that $\varphi(xA) = f(x)B$, for every $x \in G$, is an isomorphism of groups. Firstly, we prove that φ is an anti group homomorphism. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((xA)(yA)) = \varphi((xy)A) = f(xy)B = f(y)f(x)B = f(y)Bf(x)B = \varphi(yA)\varphi(xA)$$

and so φ is a group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(xA) = \varphi(yA)$, then $f(x)B = f(y)B$ and from Proposition 3.6 we get that

$$B(f(x)^{-1}f(y), q) = B(f(y)^{-1}f(x), q) = B(f(e_G), q)$$

and so

$$B(f(x^{-1})f(y), q) = B(f(y^{-1})f(x), q) = B(f(e_G), q)$$

and then

$$B(f(x^{-1}y), q) = B(f(y^{-1}x), q) = B(f(e_G), q)$$

which implies that

$$A(x^{-1}y, q) = A(y^{-1}x, q) = A(e_G, q)$$

and thus Proposition 3.6 gives us that $xA = yA$ which implies that φ is one-one. Therefore φ will be an isomorphism of groups.

4 Open Problem

In this paper, as using norms(T and C), we introduce the concepts of strongest relations, cosets and middle cosets of Q-intuitionistic fuzzy subgroups and prove some results about them. Now one can define and investigate Q-intuitionistic fuzzy subrings as we did for Q-intuitionistic fuzzy subgroups and this can be an open problem.

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References

[1] M. T. Abu Osman, *on some products of fuzzy subgroups*, Fuzzy Sets and Systems, **24** (1987), 79-86.

[2] J. M. Anthony and H. Sherwood, *Fuzzy groups redefined*, Journal of Mathematical Analysis and Application, **69** (1977), 124-130.

[3] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1986), 87-96.

[4] K. T. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61** (1994), 137-142.

[5] J. J. Buckley and E. Eslami, introduction to fuzzy logic and fuzzy sets, Springer-Verlag Berlin Heidelberg GmbH, (2002).

[6] R. Rasuli and H. Narghi, *T-Norms Over Q-Fuzzy Subgroups of Group*, Jordan Journal of Mathematics and Statistics (JJMS), **12** (1) (2019), 1-13.

[7] R. Rasuli, *Anti Q-fuzzy subgroups under T-conorms*, Earthline Journal of Mathematical Science, **4** (1) (2020), 13-28.

[8] R. Rasuli, *Conorms over conjugates and generalized characterestics of anti Q-fuzzy subgroups*, The Third national Conference on Managent and Fuzzy Systems, University of Eyvanekey, Iran, Eyvanekey , March 2021.

[9] R. Rasuli, *Conorms over level subsets and translations of anti Q-fuzzy Subgroups*, International Journal of Mathematics and Computation, **32**(2)(2021), 55-67.

[10] R. Rasuli, *Fuzzy d-algebras under t-norms*, Eng. Appl. Sci. Lett. (EASL), **5(1)**(2022), 27-36.

[11] R. Rasuli, *T-fuzzy subbigroups and normal T-fuzzy subbigroups of bi-groups*, J. of Ramannujan Society of Mathematics and Mathematical Sciences, **9(2)**(2022), 165-184.

[12] R. Rasuli, *S - (M,N)-fuzzy subgroups*, 1th National Conference on Applied Reserches in Basic Sciences(Mathematics, Chemistry and Physics) held by University of Ayatolla Boroujerdi, Iran, during May 26-27, 2022.

[13] R. Rasuli, *Intuitionistic fuzzy BCI-algebras (implicative ideals, closed implicative ideals, commutative ideals) under norms*, Mathematical Analysis and its Contemporary Applications, **4(3)**(2022), 17-34.

[14] R. Rasuli, *T-fuzzy G-submodules*, Scientia Magna, **17(1)**(2022), 107-118.

[15] R. Rasuli, *Anti Fuzzy Congruence on Product Lattices*, 7^{th} International Congerence on Combinatotcs, Crytography, Computer Science and Computing held by Iran University of Science and Technology, Iran,Tehran, during November 16-17, 2022.

[16] R. Rasuli, *t-conorms over anti fuzzy subgroups on direct product of groupos*, Annals of Mathematics and Computer Science, **10**(2022), 8-18.

[17] R. Rasuli, *Norms Over Intuitionistic Fuzzy Subgroups on Direct Product of Groups*, Commun. Combin., Cryptogr. Computer Sci., **1**(2023), 39-54.

[18] R. Rasuli, *T-norms over complex fuzzy subgroups*, Mathematical Analysis and its Contemporary Applications, **5(1)**(2023), 33-49.

[19] R. Rasuli, *T-Fuzzy subalgebras of BCI-algebras*, Int. J. Open Problems Compt. Math., **16(1)**(2023), 55-72.

[20] R. Rasuli, *Norms over Q-intuitionistic fuzzy subgroups of a group*, Notes on Intuitionistic Fuzzy Sets, **29(1)**(2023), 30-45.

[21] R. Rasuli, *Anti fuzzy multigroups and t-conorms*, 2^{th} National Conference on Soft Computing and Cognitive Science held by Conbad Kavous University, Factualy of Technology and Engineering Minutest, Iran, Minudasht, during April 18-19, 2023.

[22] R. Rasuli, *Anti fuzzy d-algebras and t-conorms*, 2^{th} National Conference on Soft Computing and Cognitive Science held by Conbad Kavous University, Factualy of Technology and Engineering Minutest, Iran, Minudasht, during April 18-19, 2023.

[23] A. Solairaju and R. Nagarajan, *A New structure and constructions of Q- fuzzy group*, Advances in Fuzzy Mathematics, **4** (2009), 23-29.

[24] X. H. Yuan and E. S. Lee, *A fuzzy algebraic system based on the theory of falling shadows*, Journal of Mathematical Analysis and Application, **208** (1977), 243-251.

[25] L. A. Zadeh, *Fuzzy sets*, Inform. Control., **8** (1965), 338-353.