

# A Note on the Chebyshev Set Problem in Normed Linear Spaces

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## Abstract

*Best approximation (BA) is an interesting field in functional analysis that has attracted a lot of attention from many researchers for a very long period of time up-to-date. Of greatest consideration is the characterization of the Chebyshev set (CS) which is a subset of a normed linear space (NLS) which contains unique BAs. However, a fundamental question remains unsolved to-date regarding the convexity of the CS in infinite NLS known as the CS problem. The question which has not been answered is: Is every CS in a NLS convex?. This question has not got any solution including the simplest form of a real Hilbert space (HS). In this note, we characterize CSs and convexity in NLSs. In particular, we consider the space of all real-valued norm-attainable functions. We show that CSs of the space of all real-valued norm-attainable functions are convex when they are closed, rotund and admits both Gateaux and Fréchet differentiability conditions.*

**Keywords:** *Nearest point, Convex set, Chebyshev set, Best approximation, Distance function.*

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## 1 Introduction

Studies in approximation theory have been carried out by many mathematicians over decades (see [1], [2], [10] and [21] and the references therein). The

most important basic question in the field of BA is the concern about the existence of BAs [7]. This is because BA theory has several applications involving finding solution to systems of equations [10]. This work is useful in contributing knowledge in functional analysis by providing at least a partial solution to the CS problem [18]. It will also be useful in solving convex optimization problems and finding solutions to differential equations [16]. Best approximation (BA) is an interesting field in functional analysis that has attracted a lot of attention from many researchers for a very long period of time up-to-date (see [3] and [4], [9] and the references there in). Of greatest consideration is the characterization of the CS which is a subset of a NLS which contains unique BAs [24]. Approximation theory involves obtaining best approximation of functions using simple functions whether they are linear or nonlinear [4], [6] and [15]. Martin [11] characterized remotality of sets with regard to normed linear spaces and in particular for convex sets in Banach spaces. However, the convexity of the Banach spaces was not done in general due to the complex nature of spaces. Mazaheri [14] also considered weakly-Chebyshev subspaces for NLSs of Banach spaces and did their characterization in terms of nearest and farthest points via distance functions. However, the authors could not give a particular best approximation for the CSs and their convexity even for the simplest case [13] of a HS of  $\ell^2$ . Zalinescu [26] on the study of convex sets and their characterizations in general spaces determined optimization criteria for vector spaces and left open a question regarding convexity of these spaces. More recently, Mazaheri and Salehi [14] studied CSs and considered conditions under which they are convex. However, they could not determine convexity of the CSs in NLSs even for the simplest set up of HSs. It is worth noting that various techniques have been used in trying to get a solution to the CS problem [4]. The first technique we discuss in this work is the Bunt-Motzkin Theorem. This theorem is a result on the converse of the CS problem. It asserts that if a set  $C$  is Chebyshev then it implies that it is convex in Bergman spaces [5]. However, this assertion still remains unknown if it holds for infinite dimensional Hilbert spaces. The other technique is the Frèchet differentiability [8]. This is a derivative defined mostly in normed spaces. Frèchet differentiability occurs on real-valued functions or vector valued functions of multiple variables. It is applicable mostly and particularly on directional derivative where the continuity of the map is essential [19]. Gateaux differentiability conditions are also very instrumental conditions in BAs. A Gateaux derivative is a fundamental principle in differential calculus which is a generalization on functions which are continuous on Banach spaces. It is useful in carrying out approximations in locally convex spaces [20]. It is useful in formalization of functional derivatives which are important in best approximations in calculus of variations. Moreover, its useful since it takes care of nonlinear functions also. Best approximation techniques have also been employed in CS problem [24]. These

are techniques in approximation theory which useful in obtaining best approximation results for functions in various spaces. They include: Polynomial approximations; Chebyshev approximations; Remez techniques and algorithms; and Pade' approximation techniques for optimal polynomials. Despite the fact that these techniques have been used in trying to solve CS problem, the answer to this problem is still elusive [23]. It is very important to unveil a detailed account of research work which have been done in approximation theory. In particular, we consider literature on conditions under which subsets of NLSs are Chebyshev. We also discuss the characterizations on distance functions of CSs in NLSs and finally give a review on investigations on convexity of CSs in various NLSs [6]. We begin with some brief account on the various studies on CSs and their subsets. CSs are important sets in approximation theory due to their properties. A lot of characterizations have been done on these sets with interesting results obtained. Their subsets are also interesting as they carry hereditary properties in them. Fletcher and Moors [7] characterized CSs and showed that a CS is particularly a subset of a NLS which has properties that helps in the establishment of best approximations results which are unique. The authors investigated characteristics of the metric projection and obtained necessary and sufficient conditions under which a a subset of a NLS becomes a CS and also conditions for a CS to satisfies convexity properties. Moreover, the authors gave an example where they constructed a nonconvex CS as shown in the next result.

**Proposition 1.1** (*[7], Proposition 2.12*) *Let  $W$  be a NLS. A subset of  $W$  which is closed is also convex if and only if it is midpoint convex.*

Proposition 1.1 characterizes NLSs in terms of midpoint convexity and shows that a subset of a NLS can be convex if its closed and satisfies midpoint convexity. However, this result does not indicate whether the space  $W$  is convex in general even if it is a CS. It is noted that closedness is useful for midpoint convexity but not for convexity in general.

Vlasov [25] in the earlier years characterized normed spaces in terms of approximate properties. Historically, it can be shown in brief, the main contributions of researchers in this field of study. In 1934 Bunt proved the convexity of a CS in the real plane. Also 1938 Kritikos followed Bunt's results and extended the theorem of Bunt to an  $n$ -dimensional real space. This was followed by the work of Efimov and Stechkin in 1961 which showed that a CS of a general HS which is approximately compact satisfies convexity condition. To conclude the history, Klee considered weak closedness and proved that CS which is weakly closed satisfies the convexity condition. This history and more details can be found in Mantegazza [9]. To consider particular cases, the author in [24] considered proximal sets which are related to Chebyshev sets and gave the result below.

**Theorem 1.2** *Let  $W$  be a NLS. Consider  $J$  as a proximal set of  $W$ . Then  $J$  is nonempty and closed.*

Theorem 1.2 considers proximal sets and characterizes closedness and the content of the sets. However, it becomes very difficult to come up with a structure of the metric projection function in terms of its geometry. Nonetheless, an exception on this assertion can be considered when  $J$  is a subspace. Next we consider distance functions of Chebyshev sets in NLSs. Distance functions are also important in characterizing CSs. By the result of Asplund [1], the CS problem was given a different dimension to consider metric projections as shown in the next proposition.

**Proposition 1.3** (*[7], Proposition 2.2*) *Let  $W$  be a NLS. Consider  $J$  as a CS of  $W$ . Then the distance function for  $J$  satisfies nonexpansivity and continuity conditions.*

Proposition 1.3 describes CSs and their distance functions in terms of continuity and nonexpansivity in a general set up. Lastly we consider CSs and convexity. We consider literature on the key question of this study, that is, Is every CS in HS convex?. We begin with the following analogy of the result of Borwein [5] on closedness, reflexivity and rotundity of Hilbert spaces.

**Lemma 1.4** (*[5], Fact 3*) *All CSs are closed and all closed sets satisfying convexity condition are Chebyshev in a rotund reflexive space. Particularly, all nonempty closed sets satisfying convexity condition in HS are Chebyshev.*

Lemma 1.4 gives an elaborate characterization of Chebyshev sets and convexity in terms of reflexivity and rotundity that requires uniqueness property however this does not answer Cs problem in general.

In summary, with all these considerations, a fundamental question remains unsolved to-date regarding the convexity of CSs in infinite NLSs known as the CS problem. The question which has not been answered is: Is every CS in a NLS convex?. This question has not been answered even in the simplest case of a real HS. In this regard, it is worth characterizing CS and convexity in NLSs. We also attempt to answer this question partially in a particular case of the NLS of all norm-attainable real-valued functions. This work is organized as follows: For the first section, we begin with a mathematical background as given in this introduction followed with the preliminary basic concepts that helps us to understand this work. We then provide the main results and finally the conclusion.

## 2 Preliminaries

For a better understanding of this work, we outline the basic definitions that are key to this note on CS and convexity in NLSs.

**Definition 2.1** ([8]) *Let  $W$  be a NLS and  $G$  be a nonempty subset of  $W$ . Consider the particular point  $\zeta \in W$ . We define the distance from the point  $\zeta$  to  $G$  by  $d(\zeta, G) = \inf_{\eta \in G} \|\zeta - \eta\|$ , and the map  $\zeta \mapsto d(\zeta, G)$  is called the distance function for  $G$ . We call  $\zeta$  the nearest point in  $G$ .*

**Definition 2.2** ([15]) *A set  $G$  in a NLS  $W$  is called a CS if every point in  $W$  has a unique nearest point in  $G$ . That is, CS is a subset of a NLS that admits unique best approximations.*

**Definition 2.3** ([8]) *Let  $D$  be a nonempty set. A subset  $E$  of  $D$  is said to be convex if for all  $\zeta, \eta \in E$  the line segment connecting  $\zeta$  and  $\eta$  is in  $E$ , that is,  $(1 - \alpha)\zeta + \alpha\eta$  is in  $E$  for  $\zeta, \eta \in E$ , and  $\alpha \in [0, 1]$ .*

At this point, we proceed to give the main results of this paper. These results are restricted to the space of all norm-attainable real-valued functions. A function  $\phi$  is said to be norm-attainable if there exists a unit vector  $\xi$  in the domain of  $\phi$  such that  $\|\phi(\xi)\| = \|\phi\|$ . The space of all norm-attainable real-valued functions is a NLS. For details on norm-attainability, see [16]-[20] and the references therein.

### 3 Main results

We provide the main results of this note in this section. We characterize CSs and their subsets and tackle the CS problem. We begin with the following proposition which considers distance functions and Gateaux differentiability. We note that all the spaces and their subspaces are all nontrivial and are strictly NLS of all functions that are norm-attainable unless otherwise stated.

**Proposition 3.1** *Let  $\mathcal{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathcal{J}$  be a closed and smooth subset of  $\mathcal{Q}$ . Let  $\zeta \in \mathcal{Q} \setminus \mathcal{J}$  and  $\eta$  the nearest point for  $\zeta$  in  $\mathcal{J}$ , then Gateaux differentiability condition of  $\mathcal{Q}$  holds for  $(\zeta - \eta)$ .*

*Proof.* Since the norm of  $\mathcal{Q}$  is Gateaux differentiable from the statement of the proposition, it suffices to prove the existence of the unique limit of the the Gateaux derivative  $\lim_{l \rightarrow 0} \frac{d_{\mathcal{J}}(\zeta + l(x - \eta)) - d_{\mathcal{J}}(\zeta)}{l}$ . From [16], we deduce that if  $l > 0$  then the limit exists. For the uniqueness, we see from the result of [7] that the limit of the derivative is unique. It follows then that  $\langle d'_{\mathcal{J}}(\zeta), x - \eta \rangle$  holds for  $d_{\mathcal{J}}(\zeta)$ . This completes the proof.

Proposition 3.1 leads to the interesting question as to what happens when  $\mathcal{Q}$  is rotund. We see this in the next lemma.

**Lemma 3.2** *Let  $\mathcal{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathcal{J}$  be a closed and smooth CS of  $\mathcal{Q}$ . Let  $\zeta \in \mathcal{Q} \setminus \mathcal{J}$  and  $\partial d_{\mathcal{J}}(\zeta)$  be a singleton set. Then the following conditions hold if the first dual of  $\mathcal{Q}$  is rotund:*

- (i).  $\phi$  on  $\mathcal{J}$  is uniformly continuous.
- (ii).  $\phi$  on  $\mathcal{J}$  is totally bounded.
- (iii).  $\mathcal{J}$  satisfies convexity condition.
- (iv).  $d_{\mathcal{J}}$  satisfies convexity condition.
- (v).  $d_{\mathcal{J}}$  satisfies Gateaux differentiability at  $\zeta$ .

*Proof.* We proceed with the proof as follows:

Case (i).  $\phi$  on  $\mathcal{J}$  is uniformly continuous since every space of norm-attainable functions contains continuous functions.

Case (ii).  $\phi$  on  $\mathcal{J}$  is totally bounded follows immediately from case (i).

Case (iii).  $\mathcal{J}$  satisfying convexity condition follows immediately from the conditions of the statement of the lemma.

Case (iv).  $d_{\mathcal{J}}$  satisfying convexity condition follows from the fact that  $\mathcal{J}$  satisfies convexity condition.

Case (v). Since  $d_{\mathcal{J}}$  satisfies convexity condition and is uniformly continuous at  $\zeta$  and from Proposition 3.1  $\partial d_K(\zeta)$  is a singleton set,  $d_{\mathcal{J}}$  satisfies Gateaux differentiability at point  $\zeta$  and we attain equality of  $d'_{\mathcal{J}}(\zeta)$  and  $\partial d_{\mathcal{J}}(\zeta)$ . This completes the proof.

At this point, we state the main theorem of our work that characterizes convexity of  $\mathcal{Q}$  in terms of Fréchet differentiability condition.

**Theorem 3.3** *Let  $\mathcal{Q}$  be a NLS of all norm-attainable real-valued functions and  $\mathcal{J}$  be a closed and smooth CS of  $\mathcal{Q}$ . Let  $\zeta \in \mathcal{Q} \setminus \mathcal{J}$  and  $\partial d_{\mathcal{J}}(\zeta)$  be a singleton set. Then  $d_{\mathcal{J}}$  satisfies Fréchet differentiability condition at  $\zeta$ .*

*Proof.* It is known from [25] that the norm of  $\mathcal{Q}$  and hence the dual norm of  $\mathcal{Q}^*$  satisfies Fréchet differentiability condition. Also from Lemma 3.2 it implies that  $\mathcal{Q}$  is strictly reflexive. Moreover,  $\mathcal{Q} \setminus \mathcal{J}$  has the nearest point  $\zeta$  and so  $\mathcal{J}$  of  $\mathcal{Q}$  satisfies Fréchet differentiability condition and so is  $\mathcal{Q}$ .

As consequences of Theorem 3.3, we state the following corollaries.

**Corollary 3.4** *Every distance function of a CS of the NLS of all norm-attainable real-valued functions is Fréchet differentiable.*

*Proof.* Follows from the conditions of Lemma 3.2 and Theorem 3.3. The rest is clear from the fact that every rotund CS is convex.

**Corollary 3.5** *Every distance function of a CS of the NLS of all norm-attainable real-valued functions is Gateaux differentiable.*

*Proof.* Follows immediately from the conditions of Theorem 3.3 and Corollary 3.5 the proof is complete.

## 4 Open Problem

In conclusion, a fundamental question that remains unsolved to-date regarding the convexity of the CS in infinite NLS known as the CS problem has been studied in this work. This CS problem which has not been solved in totality (even in this note) states that: Is every CS in a NLS convex? This question has not got any solution even in the simplest form of a real Hilbert space (HS). In this note, we have characterized Chebyshev sets and their convexity in NLSs. We considered the NLS of all real-valued norm-attainable functions. We have shown that Chebyshev subsets of the NLS of all real-valued norm-attainable functions are convex when they are closed, rotund and admits both Gateaux and Fréchet differentiability conditions. We pose an open problem for future research as: Is every CS in a NLS convex?

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