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Algorithmic Method for Approximating the Solution of the Matrix Equation $\sum_{j=1}^{m} A^{m-j}XA^{j-1} = B$

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Abstract

Matrix equations play an important place in some areas of sciences. Intensive studies aiming to solve some matrix equations and to explore some applications of these studies have been carried out recently. The contribution of the present paper falls within this framework. We investigate here an algorithm that converges to the positive solution of $\sum_{j=1}^{m} A^{m-j}XA^{j-1} = B$***, when*** A ***and*** B ***are two positive matrices. Application for some standard matrix equations, like Sylvester equation, is provided as well. Numerical examples showing the interest of this work and illustrating the theoretical study are also discussed.***

Keywords: *Algebraic Lyapunov Equations, Generalized Matrix Product, Matrices, Matrix Equations.*

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1 Introduction

We investigate here an algorithm that converges to the positive solution of

$$\sum_{j=1}^{m} A^{m-j}XA^{j-1} = B. \tag{1.1}$$

The existence of solution of the matrix equation (1.1), when A is positive invertible and B is a matrix of a special type, has been discussed by Furuta in [6]. We first write (1.1) in an equivalent form which involves the generalized matrix product recently introduced by Raïssouli and Al-Subaihi in [13]. This matrix product was a good tool for writing some matrix expressions in condensed and short forms.

A more general matrix equation, namely

$$\sum_{j=1}^{m} A^{m-j}XB^{j-1} = C,$$

was investigated by Bhatia and Uchiyama in [5], when A and B satisfy an appropriate assumption. This includes the special case of Sylvester equation [3, 10] $AX + XB = C$ and so the Lyapunov equation $AX + XA^T = C$, which are of great interest in various scientific areas, like numerical analysis, control theory, signal processing, image restoration, matrix theory and engineering.
We need more basic notions and notations. Let $n \geq 1$ be an integer. We denote by $\mathcal{M}_n$ the space of $n \times n$-matrices (with real entries) and by $\mathcal{S}_n$ the subspace of symmetric matrices in $\mathcal{M}_n$. The space $\mathcal{M}_n$ is equipped with its standard matrix norm defined by

$$\forall A \in \mathcal{M}_n \qquad \|A\| := \max\{\|Ax\|,\ \ x \in \mathbb{R}^n,\ \|x\| = 1\},$$

where $\|x\|$ is the euclidian norm of $x \in \mathbb{R}^n$.

A matrix $A \in \mathcal{M}_n$ is called positive, and we write $A \geq 0$, if $A \in \mathcal{S}_n$ and $\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{R}^n$, where $\langle ., .\rangle$ stands for the standard inner product of $\mathbb{R}^n$. The set of positive (resp. positive invertible) matrices of $\mathcal{M}_n$ will be denoted by $\mathcal{S}_n^+$ (resp. $\mathcal{S}_n^{+*}$). The positivity of matrices generates a partial order in $\mathcal{S}_n$, known as Löwner order, defined by: $A \leq B$ if and only if $A, B \in \mathcal{S}_n$ and $B - A \in \mathcal{S}_n^+$. It is well known that, for each integer $m \geq 1$ and $A \in \mathcal{S}_n^+$, the equation $X^m = A$ has one and only one solution in $X \in \mathcal{S}_n^+$, namely the m-root of A, which will be denoted by $A^{1/m}$. We say that $A \in \mathcal{M}_n$ is accretive, and we write $A \in \mathcal{A}_n$, if $A + A^T \in \mathcal{S}_n^+$, where A^T is the matrix transpose of A. We have the following chain of inclusions

$$\mathcal{S}_n^{+*} \subset \mathcal{S}_n^+ \subset \mathcal{S}_n \subset \mathcal{A}_n \subset \mathcal{M}_n.$$

The following result is known as Löwner implication,

$$\forall \alpha \in [0,1], \quad A \geq B \geq 0 \Longrightarrow A^{\alpha} \geq B^{\alpha}, \tag{1.2}$$

and the inverse implication does not, in general, hold. In particular, $A \geq B \geq 0$ implies $A^{1/2} \geq B^{1/2}$ but, $A \geq B \geq 0$ does not imply $A^2 \geq B^2$.

2 Generalized Matrix Product

Let $\mathcal{C}$ be a nonempty subset of $\mathcal{M}_n$ and $\Phi : \mathcal{C} \longrightarrow \mathcal{M}_n$ be a given map. Let $A \in \mathcal{C}$ and $B \in \mathcal{M}_n$ be such that, for $t > 0$ small enough, $A + tB \in \mathcal{C}$. We define the Φ-generalized matrix product of A and B by, [13]

$$\big[A, B\big]_{\Phi} := \lim_{t \downarrow 0} \frac{\Phi(A + tB) - \Phi(A)}{t}, \tag{2.1}$$

provided this limit exists. It is worth mentioning that, if (2.1) is well-defined then it is equivalent to

$$\Phi(A + tB) = \Phi(A) + t\big[A, B\big] + t\epsilon_t(A, B), \tag{2.2}$$

where $\epsilon_t(A, B) \to 0$ as $t \to 0$.

Some proprieties of $\big[A, B\big]_{\Phi}$ can be found in [13]. According to (1.2), it is not hard to see that

$$\forall A, B \in \mathcal{S}_n^+ \quad \forall p \geq 1 \qquad \big[A^p, B\big]_{1/p} \geq 0. \tag{2.3}$$

Example 2.1. *Let m be some integer or fractional number m. If $\Phi(X) = X^m$, we simply denote $[A, B]_{\Phi} = \big[A, B\big]_m$. It is easy to check that,*

$$[A, B]_{-1} = -A^{-1}BA^{-1}, \;\; [A, B]_2 = AB + BA,$$

and $[A, B]_{1/2}$ is solution of the Lyapunov equation $A^{1/2}X + XA^{1/2} = B$.

We also have the following result, [13]

Proposition 2.2. *Let $m \geq 2$ be an integer. The following assertions hold:*
(i) For any $A, B \in \mathcal{M}_n$ we have

$$\big[A, B\big]_m = \sum_{j=1}^{m} A^{m-j}BA^{j-1}. \tag{2.4}$$

(ii) For any $A, B \in \mathcal{S}_n^{+*}$ *we have*

$$\big[A, B\big]_{-m} = -\sum_{j=1}^{m} A^{j-m-1} B A^{-j}. \tag{2.5}$$

(iii) For any $A \in \mathcal{S}_n^{+*}$ *and* $B \in \mathcal{M}_n$ *one has*

$$\big[A, B\big]_{1/m} = BA^{\frac{1}{m}-1} - \frac{\sin\left(\frac{\pi}{m}\right)}{\pi} A \int_0^\infty s^{\frac{1}{m}-1}\big(A+sI\big)^{-1} B \big(A+sI\big)^{-1}\, ds. \tag{2.6}$$

3 Equation $\big[A, X\big]_\Phi = B$

This section discusses the resolution of the matrix equation in X, $\big[A, X\big]_\Phi = B$, and its related particular cases. We start by stating the following result.

Theorem 3.1. *Let* $\Phi : \mathcal{C} \longrightarrow \mathcal{M}_n$ *and let* $A \in \mathcal{C}$. *Assume that* Φ *is a bijection from* $\mathcal{C}$ *into* $\Phi\big(\mathcal{C}\big)$ *and* Φ^{-1} *is continuously differentiable at* $\Phi(A)$. *Then the following implication is true*

$$\big[A, X\big]_\Phi = B \Longrightarrow X = \big[\Phi(A), B\big]_{\Phi^{-1}}. \tag{3.1}$$

If moreover Φ *is continuously differentiable at* $\Phi^{-1}(A)$ *then the inverse implication of* (3.1) *holds.*

Proof. Following (2.2), with $\big[A, X\big]_\Phi = B$, we have

$$\Phi(A + tX) = \Phi(A) + tB + t\epsilon_t, \quad \text{with } \epsilon_t := \epsilon_t(A, X) \to 0 \text{ as } t \to 0.$$

Since Φ is a bijection then, composing this latter equality by Φ^{-1}, we get

$$A + tX = \Phi^{-1}\Big(\Phi(A) + tB + t\epsilon_t\Big) = A + t\nabla\Phi^{-1}\big(\Phi(A)\big)\big(B + \epsilon_t\big) + t\epsilon_t.$$

We then deduce

$$X = \nabla\Phi^{-1}\big(\Phi(A)\big)(B + \epsilon_t) + t\epsilon_t,$$

which by letting $t \to 0$ yields

$$X = \nabla\Phi^{-1}\big(\Phi(A)\big)(B) = \big[\Phi(A), B\big]_{\Phi^{-1}},$$

which concludes the proof. □

From the previous theorem we immediately deduce the following result.

Corollary 3.2. *With the same assumptions as in Theorem 3.1, if the matrix equation* $[A, X]_\Phi = B$ *has a solution in* $X \in \mathcal{M}_n$ *then this solution is unique and is given by*

$$X = \big[\Phi(A), B\big]_{\Phi^{-1}}.$$

Corollary 3.3. *Let $p \neq 0$ be a real number and $A \in \mathcal{S}_n^{+*}$. Assume that, for $B \in \mathcal{M}_n$ symmetric, the matrix equation*

$$\text{Find } X \text{ symmetric such that } \big[A, X\big]_p = B$$

has at least one solution. Then such solution is unique and is given by

$$X = \big[A^p, B\big]_{1/p}.$$

Now, let $m \geq 2$ be an integer. We will discuss the following matrix equation

$$\sum_{j=1}^{m} A^{m-j} X A^{j-1} = B, \tag{3.2}$$

We have the following main result.

Theorem 3.4. *Let $A \in \mathcal{S}_n^{+*}$ and $B \in \mathcal{S}_n$ be given. The matrix equation (3.2), in $X \in \mathcal{S}_n$, has one and only one solution given by*

$$X = \big[A^m, B\big]_{1/m} = BA^{1-m} - \frac{\sin\left(\frac{\pi}{m}\right)}{\pi} A^m \int_0^\infty s^{\frac{1}{m}-1}\big(A^m + sI\big)^{-1} B \big(A^m + sI\big)^{-1}\, ds. \tag{3.3}$$

If moreover, B is positive then so is X.

Proof. First, we mention that with our hypothesis here the matrix equation (3.2) has a unique solution, see [5]. Following (2.4), (3.2) is equivalent to $\big[A, X\big]_m = B$ and according to Corollary 3.3 we have $X = \big[A^m, B\big]_{1/m}$. The integral expression (3.3) follows from (2.6). If B is positive then X is also positive by (2.3). The proof is finished. □

4 Algorithm converging to $\big[A, B\big]_{1/m}$

Following Theorem 3.4, the positive solution of (3.2) is given by (3.3), when $A \in \mathcal{S}_n^{+*}$ and $B \in \mathcal{S}_n$. The computation of such solution X by using (3.3) is not practical and seems to be even impossible, except for few particular cases. As already pointed out before, our aim in this section is to construct an algorithm that converges to the positive solution of (3.2). For this we will first give an algorithm approximating $[A, B]_{1/m}$. We need the following lemma, see [13].

Lemma 4.1. *Let $A, B \in \mathcal{M}_n$ and $p \geq 1$ be an integer. Assume that A is invertible, then we have*

$$(A + tB)^{-p} = A^{-p} - t \sum_{i=1}^{p} A^{i-p-1} B A^{-i} + t\epsilon_t(A, B),$$

where $\epsilon_t(A, B) \to 0$ as $t \to 0$.

Let $m \geq 2$ be an integer. The author in [2], construct an iterative scheme approaching $[A,B]_{1/2}$. if $A \in \mathcal{M}_n$ is positive invertible then the m-root $A^{1/m}$ can be approximated by $X_k := X_k(A)$ defined as follows

$$X_0 = I, \ \ X_{k+1} = \frac{m-1}{m} X_k + \frac{1}{m} A X_k^{1-m}. \tag{4.1}$$

Following [2], the convergence of (X_k) to $A^{1/m}$ is quadratic and such speed of convergence can be successively accelerated to high orders $2^2, 2^3, ...$

Using (4.1), with the definition of $[A,B]_{1/m}$, we consider the matrix sequence $Y_k := Y_k(A,B,t)$ that converges to $(A+tB)^{1/m}$, for fixed $t > 0$. Following (4.1), we can then define (Y_k) by the algorithm:

$$Y_0 = I, \ \ Y_{k+1} = \frac{m-1}{m} Y_k + \frac{1}{m}(A+tB) Y_k^{1-m}. \tag{4.2}$$

From the definition of Generalized matrix product, we have the following

$$(A+tB)^{1/m} = A^{1/m} + t[A,B]_{1/m} + t\, o(t), \ \ o(t) = o(A,B,t) \to 0 \text{ as } t \to 0. \tag{4.3}$$

Then, it is natural to search Y_k in the following form

$$Y_k = X_k + t\, Z_k + t\, o_k(t), \ \ o_k(t) \to 0, \ t \to 0, \tag{4.4}$$

where $Z_k = Z_k(A,B)$ is the required matrix sequence tending towards $[A,B]_{1/m}$. Now, (4.2) with (4.4), leads to

$$Y_{k+1} = \frac{m-1}{m} X_k + \frac{m-1}{m} t Z_k + t\, o_k(t) + \frac{1}{m}(A+tB)\Big(X_k + tZ_k + t\, o_k(t)\Big)^{1-m}.$$

Thanks to Lemma 4.1 with $p = m-1 \geq 1$, we can write

$$\begin{aligned} Y_{k+1} = {} & \frac{m-1}{m} X_k + \frac{m-1}{m} t Z_k + t\, o_k(t) \\ & + \frac{1}{m}(A+tB)\Big(X_k^{1-m} - t \sum_{i=1}^{m-1} X_k^{i-m}\big(Z_k + o_k(t)\big) X_k^{-i} + t o_k(t)\Big). \end{aligned} \tag{4.5}$$

Since $X_0 = I$ and $Y_0 = I$ then (4.4) implies that we can choose $Z_0 = 0$ as initial iteration for (Z_k). Otherwise, (4.4) gives

$$Y_{k+1} = X_{k+1} + t\, Z_{k+1} + t\, o_{k+1}(t), \ \ o_{k+1}(t) \to 0, \ t \to 0. \tag{4.6}$$

By identification, (4.5) and (4.6) lead to the desired matrix sequence $Z_k = Z_k(A,B)$ defined by:

$$Z_0 = 0, \ \ Z_{k+1} = \frac{m-1}{m} Z_k + \frac{1}{m} B X_k^{1-m} - \frac{1}{m} A \sum_{i=1}^{m-1} X_k^{i-m} Z_k X_k^{-i}, \tag{4.7}$$

where $X_k := X_k(A)$ is defined by (4.1).

The fact that (X_k) converges to $A^{1/m}$ and (Y_k) converges to $(A+tB)^{1/m}$, with (4.4) and (4.3), allows us to feel that (Z_k) converges to the desired limit $[A,B]_{1/m}$.

Putting $Z := \lim Z_k$ and letting $k \uparrow \infty$ in (4.7), with the fact that $\lim X_k = A^{1/m}$, we then obtain (after simple algebraic operations)

$$Z = BA^{\frac{1-m}{m}} - A\sum_{i=1}^{m-1} A^{\frac{i-m}{m}} Z A^{\frac{-i}{m}}. \tag{4.8}$$

Remark 4.2. *It is worth mentioning that* (4.8) *corresponds to a fixed point relationship. We omit all detail about this latter point which is out of the aim of the subject of this paper.*

After this, we are in the position to resort to a sequence (T_k) that approximates the solution X of (3.2) which is given by $X = \big[A^m, B\big]_{1/m}$. For this, we define via (4.1) the sequence $V_k = V_k(A)$ as follows

$$V_0 = I, \quad V_{k+1} = \frac{m-1}{m}V_k + \frac{1}{m}A^m V_k^{1-m}. \tag{4.9}$$

According to (Z_k), the desired matrix sequence $T_k = T_k(A;B)$ may be defined as well

$$T_0 = 0, \quad T_{k+1} = \frac{m-1}{m}T_k + \frac{1}{m}BV_k^{1-m} - \frac{1}{m}A^m \sum_{i=1}^{m-1} V_k^{i-m} T_k V_k^{-i}. \tag{4.10}$$

Remark 4.3. *We mention that if* $m = 2$ *then the previous algorithms coincide with those of paper [2] . For this, we then omit to reconsider the case* $m = 2$ *in this paper. The cases* $m = 3, 4, 5$ *are itemized below.*

- For $m = 3$, the matrix equation (3.2) is given by

$$A^2X + AXA + XA^2 = B, \tag{4.11}$$

and its solution $X = \big[A^3, B\big]_{1/3}$ is approximating by the matrix sequence (T_k) such that

$$\begin{cases} V_{k+1} = \dfrac{2}{3}V_k + \dfrac{1}{3}A^3 V_k^{-2}, \\[2ex] T_{k+1} = \dfrac{2}{3}T_k + \dfrac{1}{3}BV_k^{-2} - \dfrac{1}{3}A^3\Big(V_k^{-2}T_kV_k^{-1} + V_k^{-1}T_kV_k^{-2}\Big), \end{cases}$$

with $V_0 = I$ and $T_0 = 0$.

- For $m = 4$, (3.2) becomes

$$A^3X + A^2XA + AXA^2 + XA^3 = B, \tag{4.12}$$

whose the solution $X = \left[A^4, B\right]_{1/4}$ can be approaching by (T_k) defined by

$$\begin{cases} V_{k+1} = \dfrac{3}{4}V_k + \dfrac{1}{4}A^4V_k^{-3}, \\ \\ T_{k+1} = \dfrac{3}{4}T_k + \dfrac{1}{4}BV_k^{-3} - \dfrac{1}{4}A^4\Big(V_k^{-3}T_kV_k^{-1} + V_k^{-2}T_kV_k^{-2} + V_k^{-1}T_kV_k^{-3}\Big), \end{cases}$$

with $V_0 = I$ and $T_0 = 0$.

- For $m = 5$, (3.2) is such that

$$A^4X + A^3XA + A^2XA^2 + AXA^3 + XA^4 = B, \tag{4.13}$$

with solution $X = \left[A^5, B\right]_{1/5}$ is limit of the matrix sequence (T_k) given by

$$\begin{cases} V_{k+1} = \dfrac{4}{5}V_k + \dfrac{1}{5}A^5V_k^{-4}, \\ \\ T_{k+1} = \dfrac{4}{5}T_k + \dfrac{1}{5}BV_k^{-4} - \dfrac{1}{5}A^5\Big(V_k^{-4}T_kV_k^{-1} + V_k^{-3}T_kV_k^{-2} + V_k^{-2}T_kV_k^{-3} + V_k^{-1}T_kV_k^{-4}\Big), \end{cases}$$

with $V_0 = I$ and $T_0 = 0$.

5 Numerical examples

In order to illustrate the significance of the previous algorithms we present the following numerical example.

Example 5.1. *Let A and B be the following matrices:*

$$A = \begin{pmatrix} 13 & 4 & -5 \\ 4 & 17 & 2 \\ -5 & 2 & 19 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 24 & 10 & -5 \\ 10 & 30 & -8 \\ -5 & -8 & 55 \end{pmatrix}.$$

Executing MATLAB 2021a with 3500 digits floating arithmetic we obtain the following results which approximate the solutions of the matrix equations (4.11), (4.12) *and* (4.13)*, respectively. See TABLE* 1.

The following example discusses the numerical stability of the previous algorithms and shows that slight perturbing matrices of A and B affect small perturbations on the solution of (3.2).

Table 1: Estimations of Example 5.1.

	m=3		m=4		m=5	
	$\|V_{k+1} - V_k\|_2$	$\|T_{k+1} - T_k\|_2$	$\|V_{k+1} - V_k\|_2$	$\|T_{k+1} - T_k\|_2$	$\|V_{k+1} - V_k\|_2$	$\|T_{k+1} - T_k\|_2$
k	Alg. (4.9)	Alg. (4.10)	Alg. (4.9)	Alg. (4.10)	Alg. (4.9)	Alg. (4.10)
1	3475.9	19.699	56960	14.774	995580	11.819
10	45.072	0.25729	1425.6	0.36977	33406	0.39659
18	4.4225e-8	5.1812e-9	142.73	0.38052e-2	5604.6	0.66537e-2
19	8.9523e-17	2.0821e-17	107.04	0.28556e-2	4483.7	0.53229e-2
20	3.6682e-34	1.7e-34	80.282	0.20475e-2	3586.9	0.42583e-2
21	6.159e-69	5.6981e-69	60.209	0.15106e-2	2869.6	0.34067e-2
22	1.7362e-138	3.2097e-138	45.151	0.11323e-2	2295.6	0.27253e-2
23	1.3798e-277	5.0991e-277	33.85	0.84981e-3	1836.5	0.21803e-2
24	8.7138e-556	6.4391e-555	25.356	0.63877e-3	1469.2,	0.17443e-2
25	3.4754e-1112	5.1358e-1111	18.943	0.48242e-3	1175.4	0.13955e-2
26	5.5286e-2225	1.6339e-2223	14.034	0.36961e-3	940.3	0.11165e-2
27	2.0524e-3501	5.3586e-3502	10.126	0.29444e-3	752.24	0.89354e-3
35	-	-	1.8311e-36	6.5235e-38	126.2	0.14461e-3
36	-	-	2.3019e-73	1.6389e-74	100.96	0.11568e-3
37	-	-	3.638e-147	5.1781e-148	80.771	0.92547e-4
38	-	-	9.0869e-295	2.5862e-295	64.617	0.74038e-4
39	-	-	5.6692e-590	3.2266e-590	51.693	0.59231e-4
40	-	-	2.2066e-1180	2.5116e-1180	41.354	0.47386e-4
41	-	-	3.3429e-2361	7.6099e-2361	33.083	0.37912e-4
42	-	-	3.5469e-3494	3.1464e-3495	26.464	0.30336e-4
50	-	-	-	-	2.1047	0.84693e-5
51	-	-	-	-	0.47655	0.39315e-5
52	-	-	-	-	0.021688	3.5992e-6
53	-	-	-	-	0.43145e-5	1.4238e-8
54	-	-	-	-	1.7041e-10	1.121e-13
55	-	-	-	-	2.6583e-21	3.4916e-24
56	-	-	-	-	6.4689e-43	1.698e-45
57	-	-	-	-	3.8307e-86	2.0102e-88
58	-	-	-	-	1.3433e-172	1.4096e-174
59	-	-	-	-	1.6519e-345	3.4664e-347
60	-	-	-	-	2.4981e-691	1.0483e-692
61	-	-	-	-	5.7126e-1383	4.7945e-1384
62	-	-	-	-	2.9873e-2766	5.0144e-2767
63	-	-	-	-	1.8994e-3484	2.4393e-3486

Table 2: Estimations of Example 5.2.

	m=3		m=4		m=5	
	$\|V_{k+1}-V_k\|_2$	$\|T_{k+1}-T_k\|_2$	$\|V_{k+1}-V_k\|_2$	$\|T_{k+1}-T_k\|_2$	$\|V_{k+1}-V_k\|_2$	$\|T_{k+1}-T_k\|_2$
k	Alg. (4.9)	Alg. (4.10)	Alg. (4.9)	Alg. (4.10)	Alg. (4.9)	Alg. (4.10)
1	3381	19.697	54882	14.773	950040	11.818
10	43.837	0.25256	1373.6	0.36974	31878	0.39655
18	1.9534e-8	2.377e-9	137.52	0.37479e-2	5348.3	0.6653e-2
19	1.7086e-17	4.1325e-18	103.14	0.27058e-2	4278.6	0.53224e-2
20	1.3071e-35	6.3039e-36	77.353	0.20146e-2	3422.9	0.42579e-2
21	7.6499e-72	7.3679e-72	58.013	0.15109e-2	2738.3	0.34064e-2
22	2.6204e-144	5.0437e-144	43.504	0.11336e-2	2190.7	0.27251e-2
23	3.0745e-289	1.1831e-288	32.614	0.85118e-3	1752.5	0.21802e-2
24	4.2324e-579	3.2568e-578	24.429	0.64072e-3	1402	0.17445e-2
25	8.0207e-1159	1.2343e-1157	18.246	0.48612e-3	1121.6	0.13963e-2
26	2.8805e-2318	8.865e-2317	13.508	0.37783e-3	897.29	0.11187e-2
27	5.5024e-3500	1.4073e-3500	9.7211	0.30896e-3	717.83	0.89914e-3
35	-	-	8.4075e-40	3.3036e-41	120.43	0.14481e-3
36	-	-	4.7476e-80	3.7286e-81	96.346	0.11585e-3
37	-	-	1.5138e-160	2.3771e-161	77.077	0.92678e-4
38	-	-	1.5392e-321	4.8332e-322	61.662	0.74143e-4
39	-	-	1.5912e-643	9.9924e-644	49.329	0.59316e-4
40	-	-	1.7006e-1287	2.1358e-1287	39.463	0.47456e-4
41	-	-	1.9425e-2575	4.8789e-2575	31.57	0.37973e-4
42	-	-	3.9041e-3491	5.2616e-3492	25.254	0.304e-4
50	-	-	-	-	1.705	0.8132e-5
51	-	-	-	-	0.29773	0.29209e-5
52	-	-	-	-	0.81488e-3	1.6023e-6
53	-	-	-	-	5.9509e-6	2.3298e-9
54	-	-	-	-	3.1714e-12	2.4773e-15
55	-	-	-	-	9.0067e-25	1.4055e-27
56	-	-	-	-	7.2645e-50	2.266e-52
57	-	-	-	-	4.726e-100	2.9474e-102
58	-	-	-	-	2.0001e-200	2.4944e-202
59	-	-	-	-	3.5825e-401	8.9352e-403
60	-	-	-	-	1.1493e-802	5.733e-804
61	-	-	-	-	1.183e-1605	1.1801e-1606
62	-	-	-	-	1.2532e-3211	2.5003e-3212
63	-	-	-	-	6.4238e-3480	2.0848e-3481

Example 5.2. *Let $\tilde{A}$ and $\tilde{B}$ be the perturbing matrices of the previous matrices A and B, respectively:*

$$\tilde{A} = \begin{pmatrix} 12.0022 & 3.8888 & -5.0000 \\ 4.0000 & 16.0222 & 2.0000 \\ -5.0000 & 3.0002 & 19.0000 \end{pmatrix} \quad \textit{and} \quad \tilde{B} = \begin{pmatrix} 24.0000 & 10.0003 & -5.0000 \\ 9.8877 & 30.0222 & -8.0000 \\ -5.0000 & -8.0000 & 55.0001 \end{pmatrix}.$$

We obtain the estimations presented in TABLE 2.

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