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Power bounded generalized Volterra companion operators on Fock spaces

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Abstract

We characterize power bounded generalized Volterra companion operator on the Fock space, in terms of functional-analytic properties of the inducing functions. Moreover, we describe Ritt's resolvent growth condition for a bounded generalized Volterra companion operator.

Keywords: *Fock spaces, Generalized Volterra companion operator, Power bounded, Ritt's resolvent condition, Kreiss resolvent condition.*

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1 Introduction

Let T be a continuous self map linear operator on a Banach space $\mathcal{X}$. One of the classical operator theoretic problems for T is to identify the relation between the size of the resolvent $(T-\lambda I)^{-1}$ when λ is near to the spectrum of T and the growth of $\|T^n\|$, $n \in \mathbb{N}$, where T^n denotes the n^{th} iterate of T, that is the operator T is composed to itself n times, and $I = T^0$ is identity map. An early study to the growth of $\|T^n\|$ was made by Ritt [14] with the following condition (1), called Ritt's resolvent condition,

$$\|(T-\lambda I)^{-1}\| \leq \frac{C}{|\lambda - 1|},\ |\lambda| > 1, \tag{1}$$

where C is a given constant. Later in [5], Kreiss studied under the assumption that T satisfies the following condition (2), called Kreiss resolvent condition,

$$\|(T-\lambda I)^{-1}\| \leq \frac{C}{|\lambda| - 1},\ |\lambda| > 1. \tag{2}$$

Since Ritt's resolvent condition is stronger than Kreiss resolvent condition, any operator which satisfies (1) satisfies also (2). But the converse in general is not true (see for example [2]). Both Ritt and Kreiss resolvent conditions have an application in the stability analysis of numerical processes for solving initial value problems [2].

A result in [14] shows that, if an operator T satisfies (1), then $\frac{\|T^n\|}{n}$ goes to zero as $n \to \infty$. This is also the case for the Kreiss resolvent condition if the space $\mathcal{X}$ is infinite dimensional (see [17]). On the other hand, if the operator T is power bounded, that is there exists a positive constant C such that $\|T^n\| \leq C$, $\forall n \in \mathbb{N}$, then T satisfies the condition in (2). Whereas, power boundedness of T in general does not imply the condition in (1). But, a result in [13], obtained by Nagy and Zemanek, and independently by Lyubich [8], shows that T satisfies the condition (1) if and only if it is power bounded and

$$\sup_{n\geq 1} n\|T^{n+1} - T^n\| < \infty. \tag{3}$$

Power boundedness and Ritt's resolvent condition have been investigated for different operators acting on different Banach space of analytic functions. We mention some works on the Fock type space. Seyoum, Mengestie and Bonet [16] studied power bounded and other dynamical properties of composition operators on the generalized Fock space, showing that a bounded composition operator is power bounded. The result in [10] also shows that a non identity bounded composition operator on the Fock space satisfies Ritt's resolvent condition if and only if it is compact. One can see the material in [15] for similar works for weighted composition operators. Recently in [1], Bonet, Mengestie and the second author of this paper studied different dynamical properties, including power boundedness and Ritt's resolvent condition, for differentiation, Hardy and Volterra-type integral operators on the generalized Fock space with weight function $|z|^m, m > 0$.

In the present paper, we study power bounded and Ritt's resolvent condition for generalized Volterra companion operators on the Fock space.

2 Preliminaries

Let g and φ be entire functions on the complex plane $\mathbb{C}$. Then the generalized Volterra companion operator, $J_{(g,\varphi)}$, induced by g and φ is defined by

$$J_{(g,\varphi)}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)\,d\zeta. \tag{4}$$

In particular, when $\varphi(\zeta) = \zeta$, $J_{(g,\varphi)}$ is just the Volterra companion operator,

$$J_g f(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta.$$

The operator was first introduced by Li and Stević in [6, 7], and has found an application in the study of linear isometries of spaces of analytic functions [4]. Li and Stević studied bounded and compact property of the operator acting between space of analytic functions on the unit disk. Later in [9], Mengestie has considered the operator on Fock spaces.

It is easy to see from (4) that the n^{th} iterate of $J_{(g,\varphi)}$, by induction, is given by

$$J^n_{(g,\varphi)}f(z) = \int_0^z \big(\prod_{i=0}^{n-1} g(\varphi^i(\zeta))\big) f'(\varphi^n(\zeta))d\zeta,$$

for each positive integer n, where

$$\varphi^i = \underbrace{\varphi \circ \varphi \circ \varphi \circ \cdots \circ \varphi}_{i-\text{times}}$$

and φ^0 is the identity map on the complex plane $\mathbb{C}$. Thus, $J^n_{(g,\varphi)}$ itself is a generalized Volterra companion operator induced by the pair of entire functions (g_n, φ^n), where $g_n(\zeta) := \prod_{i=0}^{n-1} g(\varphi^i(\zeta))$ and φ^n, for $\varphi(\zeta) = a\zeta + b$ with $|a| \le 1$, is

$$\varphi^n(\zeta) = \begin{cases} a^n\zeta + \frac{b(1-a^n)}{1-a}, & a \neq 1 \\ \zeta + nb, & a = 1. \end{cases}$$

We next recall that the Fock space F_p, $1 \le p \le \infty$, is the space of entire functions f such that the norm given by

$$\|f\|_{F_p} := \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} dm(\zeta)\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \sup_{\zeta\in\mathbb{C}} |f(\zeta)| e^{-\frac{|\zeta|^2}{2}}, & p = \infty \end{cases}$$

is finite, where dm denotes the Lebesgue area measure. This norm can be expressed in terms of the Littlewood-Paley type derivative formula (see [3, 11]) as follows:

$$\|f\|_{F_p} \simeq \begin{cases} \left(|f(0)|^p + \int_{\mathbb{C}} |f'(\zeta)|^p (1+|\zeta|)^{-p} e^{-\frac{p}{2}|\zeta|^2} dm(\zeta)\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ |f(0)| + \sup_{\zeta\in\mathbb{C}} |f'(\zeta)|(1+|\zeta|)e^{-|\zeta|^2/2}, & p=\infty, \end{cases} \tag{5}$$

where the notation $A(\zeta) \simeq B(\zeta)$ means both $A(\zeta) \lesssim B(\zeta)$ and $B(\zeta) \lesssim A(\zeta)$, where $A(\zeta) \lesssim B(\zeta)$ (or equivalently $B(\zeta) \gtrsim A(\zeta)$) means that there is a constant α such that $A(\zeta) \le \alpha B(\zeta)$, for each $\zeta \in \mathbb{C}$. Let K_w be a kernel function for F_2, defined by $K_w(\zeta) = e^{\overline{w}\zeta}$ and belongs to all Fock spaces F_p with $\|K_w\|_{F_p} = e^{\frac{|w|^2}{2}}$, and k_w be the normalized function

$$k_w(\zeta) = \frac{K_w(\zeta)}{\|K_w\|_{F_p}} = e^{\overline{w}\zeta - \frac{|w|^2}{2}}.$$

3 Main results

We begin the section with the following useful lemma and proposition from [12] for further use. For simplicity, we define

$$\mathcal{N}_{(g,\varphi)}(\zeta) := \frac{|g(\zeta)||\varphi(\zeta)|}{1+|\zeta|} e^{\frac{1}{2}(|\varphi(\zeta)|^2 - |\zeta|^2)}$$

and $\mathcal{M}_{(g,\varphi)} := \sup_{\zeta\in\mathbb{C}} \mathcal{N}_{(g,\varphi)}(\zeta)$.

Lemma 3.1. *If $\mathcal{N}_{(g,\varphi)}$, where φ is nonconstant, is bounded on the complex plane $\mathbb{C}$, then $\varphi(\zeta) = a\zeta + b$ with $|a| \le 1$. In particular, if $|a| = 1$, then g has the form*

$$g(\zeta) = g(0)K_{-\overline{a}b}(\zeta). \tag{6}$$

Proposition 3.2. *Let $1 \le p \le \infty$ and φ be nonconstant. Then the generalized Volterra companion operator $J_{(g,\varphi)} : F_p \to F_p$ is bounded if and only if $\mathcal{M}_{(g,\varphi)}$ is finite. Moreover,*

$$\mathcal{M}_{(g,\varphi)} \lesssim \|J_{(g,\varphi)}\| \lesssim |a|^{-\frac{2}{p}} \mathcal{M}_{(g,\varphi)}, \tag{7}$$

where $\varphi(\zeta) = a\zeta + b$ with $|a| \le 1$.

We may now state our first main result. Note that, if the inducing symbol function g of $J_{(g,\varphi)}$ is identically zero, then clearly $J_{(g,\varphi)}$ becomes a zero operator for which many properties hold trivially. Hence, we assume that g is not identically zero in the remaining part of the manuscript.

Theorem 3.3. *Let* $1 \leq p \leq \infty$ *and* $J_{(g,\varphi)} : F_p \to F_p$ *be bounded. Then* $\varphi(\zeta) = a\zeta + b$ *with* $|a| \leq 1$.

(I) For $|a| = 1$;

- *(a) If* $b = 0$, *then* $J_{(g,\varphi)}$ *is power bounded if and only if* g *is constant,* $g(\zeta) = g(0)$, *with* $|g(0)| \leq 1$.
- *(b) If* $a \neq 1$ *and* $b \neq 0$, *then* $J_{(g,\varphi)}$ *is power bounded if and only if* $|g(0)| \leq e^{-\frac{|b|^2}{2}}$.
- *(c) If* $a = 1$ *and* $b \neq 0$, *then* $J_{(g,\varphi)}$ *is power bounded if and only if* $|g(0)| < e^{-\frac{|b|^2}{2}}$.

(II) For $0 < |a| < 1$;

- *(a) If* $J_{(g,\varphi)}$ *is power bounded, then* $|g(\frac{b}{1-a})| \leq 1$.
- *(b) If* g *is constant,* $g(\zeta) = g(0)$, *with*

$$|g(0)| \leq \begin{cases} |a|^{\frac{-2}{p}}, & p < \infty \\ 1, & p = \infty, \end{cases} \tag{8}$$

then $J_{(g,\varphi)}$ *is power bounded.*

Proof. (I) (a) From equation (6) and the condition that $b = 0$, we have $g(\zeta) = g(0)K_{-\overline{a}b}(\zeta) = g(0)$, $e^{\frac{1}{2}(|\varphi^n(\zeta)|^2 - |\zeta|^2)} = 1$ and $g_n(\zeta) = (g(0))^n$. Using this, we obtain

$$\left(\frac{|\zeta_0|}{1 + |\zeta_0|}\right)|g(0)|^n \leq \mathcal{M}_{(g_n,\varphi^n)} \leq |g(0)|^n,$$

where ζ_0 is a fixed nonzero complex number. From this and Proposition 3.2 the conclusion follows easily.

(b) From [15], we have $e^{\frac{1}{2}(|\varphi^n(\zeta)|^2 - |\zeta|^2)} = e^{\mathcal{R}e(a^n\overline{b}\frac{(1-\overline{a}^n)}{1-\overline{a}}\zeta) + \frac{|b\frac{(1-a^n)}{1-a}|^2}{2}}$ and

$$|g_n(\zeta)| = |g(0)|^n e^{n\frac{|b|^2}{2}} e^{-\mathcal{R}e\left(a\overline{b}\frac{(1-a^n)}{1-a}\zeta\right)} e^{\mathcal{R}e\left(\frac{a|b|^2(1-a^n)}{(1-a)^2}\right)}.$$

Thus, using the fact that $a^n\overline{b}\frac{(1-\overline{a}^n)}{1-\overline{a}}\zeta = a\overline{b}\frac{(1-a^n)}{1-a}\zeta$ for $|a| = 1$ (see [15]), we get

$$|g_n(\zeta)|e^{\frac{1}{2}(|\varphi^n(\zeta)|^2 - |\zeta|^2)} = |g(0)|^n e^{n\frac{|b|^2}{2}} e^{\mathcal{R}e\left(\frac{a|b|^2(1-a^n)}{(1-a)^2}\right) + \frac{|b\frac{(1-a^n)}{1-a}|^2}{2}}.$$

From this and the following inequalities,

$$e^{-\frac{2|b|^2}{|1-a|^2}} \leq e^{\mathcal{R}e\left(\frac{a|b|^2(1-a^n)}{(1-a)^2}\right) + \frac{|b\frac{(1-a^n)}{1-a}|^2}{2}} \leq e^{\frac{2|b|^2}{|1-a|^2} + |b|^2\frac{2}{|1-a|^2}}$$

and $\frac{|\frac{b}{1-a}|}{1+|\frac{b}{1-a}|} \leq \sup_{\zeta\in\mathbb{C}} \frac{|a^n\zeta+\left(\frac{b(1-a^n)}{1-a}\right)|}{1+|\zeta|} \lesssim \frac{|b|}{|1-a|}$, we obtain $\mathcal{M}_{(g_n,\varphi^n)} \simeq |g(0)|^n e^{n\frac{|b|^2}{2}}$. By Proposition 3.2 and estimate (7) we conclude that $J_{(g,\varphi)}$ is power bounded if and only if $|g(0)| \leq e^{-\frac{|b|^2}{2}}$.

(c) Similar procedures as in the above proof gives the following estimates,

$$e^{\frac{1}{2}(|\varphi^n(\zeta)|^2-|\zeta|^2)} = e^{\frac{1}{2}(|\zeta+nb|^2-|\zeta|^2)} = e^{n\mathcal{R}e(\bar{b}\zeta)+n^2\frac{|b|^2}{2}},$$

$$\begin{aligned}|g_n(\zeta)| = |g(0)|^n|\prod_{i=0}^{n-1} e^{-\bar{b}(\zeta+ib)}| &= |g(0)|^n|e^{-\bar{b}\sum_{i=0}^{n-1}(\zeta+ib)}| \\ &= |g(0)|^n e^{-n\mathcal{R}e(\bar{b}\zeta)-n^2\frac{|b|^2}{2}+n\frac{|b|^2}{2}} \qquad (9)\end{aligned}$$

and

$$n|b| \leq \sup_{\zeta\in\mathbb{C}} \frac{|\zeta+nb|}{1+|\zeta|} \leq \sup_{\zeta\in\mathbb{C}} \frac{|\zeta|}{1+|\zeta|} + \sup_{\zeta\in\mathbb{C}} \frac{|nb|}{1+|\zeta|} \leq 1+n|b|.$$

Combining the above estimates, we get $\mathcal{M}_{(g_n,\varphi^n)} \simeq n|g(0)|^n e^{n\frac{|b|^2}{2}}$, and hence by Proposition 3.2, $J_{(g,\varphi)}$ is power bounded if and only if $|g(0)| < e^{-\frac{|b|^2}{2}}$.

(II) (a) If $J_{(g,\varphi)}$ is power bounded, then by the estimate in (7), $\mathcal{M}_{(g_n,\varphi^n)} < \infty$ and hence $\infty > \mathcal{M}_{(g_n,\varphi^n)} \gtrsim |g(\frac{b}{1-a})|^n$. Therefore, $|g(\frac{b}{1-a})| \leq 1$.

(b) Since $\varphi(\zeta) = a\zeta+b$ with $|a| < 1$, we have $|\varphi^n(\zeta)| \lesssim |\zeta|$ and hence $e^{\frac{1}{2}(|\varphi^n(\zeta)|^2-|\zeta|^2)}$ is bounded. Moreover, the function $\frac{|\varphi^n(\zeta)|}{1+|\zeta|}$ is bounded and $|g_n(\zeta)| = |g(0)|^n$. Thus, $\mathcal{M}_{(g_n,\varphi^n)} \lesssim |g(0)|^n$. Since $|g(0)| \leq \begin{cases} |a|^{\frac{-2}{p}}, & p<\infty \\ 1, & p=\infty, \end{cases}$ by Proposition 3.2 we conclude $J_{(g,\varphi)}$ is power bounded. □

Corollary 3.4. *Let $1 \leq p \leq \infty$. Then $J_g : F_p \to F_p$ is power bounded if and only if g is constant, $g(\zeta) = g(0)$, with $|g(0)| \leq 1$.*

Theorem 3.5. *Let $1 \leq p \leq \infty$ and $J_{(g,\varphi)}$ be bounded on F_p and hence $\varphi(\zeta) = a\zeta+b$, $|a| \leq 1$.*

- *(a) If $|a| = 1$ and $|g(0)| < e^{-\frac{|b|^2}{2}}$, then $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition.*
- *(b) If $0 < |a| < 1$ and g is constant satisfying the strict version of the inequality in (8), then $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition.*

(c) If $a \neq 1$ *and* $J_{(g,\varphi)}$ *satisfies the Ritt's resolvent condition, then one of the following holds;*

(i) $b \neq 0$ *and* $\big|g(\frac{b}{1-a})\big| \leq 1$.

(ii) $b = 0$ *and* $\big|g(0)\big| \leq 1$.

(d) If $a = 1$ *and* $J_{(g,\varphi)}$ *satisfies the Ritt's resolvent condition, then one of the following holds;*

(i) $b \neq 0$ *and* $|g(0)| = 1$.

(ii) $b \neq 0$ *and* $\big|g(0)\big| < e^{-\frac{|b|^2}{2}}$.

(iii) $b = 0$ *and* $|g(0)| \leq 1$.

Proof. (a) By Theorem 3.3, $J_{(g,\varphi)}$ is power bounded. Thus, it is enough to show that (3) is satisfied.

$$\sup_{n\geq 1} n\|J^{n+1}_{(g,\varphi)} - J^{n}_{(g,\varphi)}\| \lesssim \sup_{n\geq 1} n\beta_{n+1} + \sup_{n\geq 1} n\beta_n < \infty,$$

where

$$\beta_n = \begin{cases} n|g(0)|^n e^{\frac{n|b|^2}{2}}, & a = 1 \text{ and } b \neq 0 \\ |g(0)|^n e^{\frac{n|b|^2}{2}}, & \text{otherwise.} \end{cases}$$

Therefore, $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition.

(b) The proof follows from Theorem 3.3 and by the same procedure as above.
(c) Suppose $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition, then for $b \neq 0$, using the formula in 5,

$$\begin{aligned}
\infty > n\|J^{n+1}_{(g,\varphi)} - J^{n}_{(g,\varphi)}\| &\gtrsim n\|J^{n+1}_{(g,\varphi)}k_w - J^{n}_{(g,\varphi)}k_w\|_{F_p} \gtrsim n\|J^{n+1}_{(g,\varphi)}k_w - J^{n}_{(g,\varphi)}k_w\|_{F_\infty} \\
&\simeq n \sup_{\zeta\in\mathbb{C}} \frac{|\overline{w} g_{n+1}(\zeta) e^{\varphi^{n+1}(\zeta)\overline{w} - \frac{|w|^2}{2}} - \overline{w} g_n(\zeta) e^{\varphi^{n}(\zeta)\overline{w} - \frac{|w|^2}{2}}|}{1+|\zeta|} e^{-\frac{|\zeta|^2}{2}} \\
&= n \sup_{\zeta\in\mathbb{C}} \frac{|\overline{w}\prod_{i=0}^{n} g(\varphi^i(\zeta)) e^{\varphi^{n+1}(\zeta)\overline{w} - \frac{|w|^2}{2}} - \overline{w}\prod_{i=0}^{n-1} g(\varphi^i(\zeta)) e^{\varphi^{n}(\zeta)\overline{w} - \frac{|w|^2}{2}}|}{1+|\zeta|} e^{-\frac{|\zeta|^2}{2}} \\
&= n \sup_{\zeta\in\mathbb{C}} \left(|g(\varphi^n(\zeta)) e^{\varphi^{n+1}(\zeta)\overline{w}} - e^{\varphi^{n}(\zeta)\overline{w}}| \right) \left(\frac{|w||\prod_{i=0}^{n-1} g(\varphi^i(\zeta))|}{1+|\zeta|} e^{-\frac{|w|^2}{2} - \frac{|\zeta|^2}{2}} \right) \\
&\geq n\bigg(|g(\frac{b}{1-a})| - 1 \bigg) \left(\frac{|w||g(\frac{b}{1-a})|^n |e^{\overline{w}(\frac{b}{1-a})}|}{1+|\frac{b}{1-a}|} e^{-\frac{|w|^2}{2} - \frac{|\frac{b}{1-a}|^2}{2}} \right) \\
&\geq n\Big|\frac{b}{1-a}\Big| \bigg(|g(\frac{b}{1-a})| - 1 \bigg) |g(\frac{b}{1-a})|^n.
\end{aligned}$$

The last above estimates are by putting $w = \zeta = \frac{b}{1-a}$ and the conclusion for $b \neq 0$ is obtained from that. The conclusion for the case $b = 0$ are obtained putting $\zeta = 0$ and $w = 1$ in the above last two estimates.

(d) Following similar procedure as above, we have

$$\begin{aligned}\infty &> n\|J_{(g,\varphi)}^{n+1} - J_{(g,\varphi)}^{n}\| \\ &\gtrsim n\Bigg(|g(\varphi^n(\zeta))e^{\varphi^{n+1}(\zeta)\overline{w}} - e^{\varphi^n(\zeta)\overline{w}}|\Bigg)\Bigg(\frac{|w||\prod_{i=0}^{n-1} g(\varphi^i(\zeta))|}{1+|\zeta|}e^{-\frac{|w|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg) \\ &\geq n\Bigg(|g(\varphi^n(\zeta))e^{\varphi^{n+1}(\zeta)\overline{w}}| - |e^{\varphi^n(\zeta)\overline{w}}|\Bigg)\Bigg(\frac{|w||\prod_{i=0}^{n-1} g(\varphi^i(\zeta))|}{1+|\zeta|}e^{-\frac{|w|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg),\end{aligned} \tag{10}$$

for all $\zeta, w \in \mathbb{C}$. For $b \neq 0$, putting $w = \varphi^n(\zeta)$ and using the estimates in (6), (9) and $\varphi^{n+1}(\zeta) = \varphi^n(\zeta) + b$ for $a = 1$, (10) is equal to

$$\begin{aligned}n&\Bigg(|g(\varphi^n(\zeta))e^{\varphi^{n+1}(\zeta)\overline{\varphi^n(\zeta)}}| - |e^{\varphi^n(\zeta)\overline{\varphi^n(\zeta)}}|\Bigg)\Bigg(\frac{|\varphi^n(\zeta)||\prod_{i=0}^{n-1} g(\varphi^i(\zeta))|}{1+|\zeta|}e^{-\frac{|\varphi^n(\zeta)|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg) \\ &= n\Bigg(|g(\varphi^n(\zeta))e^{b\overline{\varphi^n(\zeta)}}| - 1\Bigg)\Bigg(\frac{|\varphi^n(\zeta)||\prod_{i=0}^{n-1} g(\varphi^i(\zeta))|}{1+|\zeta|}e^{\frac{|\varphi^n(\zeta)|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg) \\ &= n\Bigg(|g(0)e^{-\overline{b}\varphi^n(\zeta)+b\overline{\varphi^n(\zeta)}}| - 1\Bigg) \\ &\qquad \times\Bigg(\frac{|\varphi^n(\zeta)||g(0)|^n e^{-n\mathcal{R}e(\overline{b}\zeta)-n^2\frac{|b|^2}{2}+n\frac{|b|^2}{2}}}{1+|\zeta|}e^{\frac{|\varphi^n(\zeta)|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg) \\ &= n\Bigg(|g(0)| - 1\Bigg)\Bigg(\frac{|\varphi^n(\zeta)||g(0)|^n e^{-n\mathcal{R}e(\overline{b}\zeta)-n^2\frac{|b|^2}{2}+n\frac{|b|^2}{2}}}{1+|\zeta|}e^{\frac{|\varphi^n(\zeta)|^2}{2}-\frac{|\zeta|^2}{2}}\Bigg).\end{aligned}$$

Putting in particular $\zeta = 0$ and $\varphi^n(0) = nb$, we obtain

$$\infty > n^2|b|\Bigg(|g(0)| - 1\Bigg)|g(0)|^n e^{n\frac{|b|^2}{2}},$$

which yields the conclusion for the case $b \neq 0$. For $b = 0$, that is $\varphi(\zeta) = \zeta$, from (10) and putting $w = 1$ and $\zeta = 0$, we obtain

$$\infty > n\Bigg(|g(0)| - 1\Bigg)|g(0)|^n.$$

Therefore, $|g(0)| \leq 1$ in this case. □

We remark that, if φ is constant, then it is easy to see from Proposition 3.2 that $J_{(g,\varphi)} : F_p \to F_p$, $1 \leq p \leq \infty$, is bounded if and only if the function

$\frac{|g(\zeta)|}{1+|\zeta|}e^{\frac{-|\zeta|^2}{2}}$ is uniformly bounded over $\mathbb{C}$. Moreover, $\|J_{(g,\varphi)}\| \simeq \sup_{\zeta\in\mathbb{C}} \frac{|g(\zeta)|}{1+|\zeta|}e^{\frac{-|\zeta|^2}{2}}$. In this case, we have also $J^n_{(g,\varphi)}f(z) = \int_0^z f'(b)(g(b))^{n-1}g(\zeta)d\zeta = J_{(g_n,\varphi)}f(z)$, where $\varphi(\zeta) = b$ and $g_n(\zeta) := (g(b))^{n-1}g(\zeta)$. Therefore, $J_{(g,\varphi)} : F_p \to F_p$, $1 \le p \le \infty$, is power bounded if and only if

$$\sup_{n\in\mathbb{N}}\sup_{\zeta\in\mathbb{C}} \frac{|g_n(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}} = \sup_{n\in\mathbb{N}}\sup_{\zeta\in\mathbb{C}} \frac{|g(b)|^{n-1}|g(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}} < \infty. \tag{11}$$

Proposition 3.6. *Let $1 \le p \le \infty$ and $J_{(g,\varphi)} : F_p \to F_p$ be bounded, where $\varphi(\zeta) = b$.*

(a) $J_{(g,\varphi)}$ is power bounded if and only if $|g(b)| \le 1$.

(b) If $g(b) = 1$, then $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition if and only if it is power bounded.

(c) If $g(b) \neq 1$, then $J_{(g,\varphi)}$ satisfies the Ritt's resolvent condition if and only if $|g(b)| < 1$.

Proof. (a) If $J_{(g,\varphi)}$ is power bounded, then from (11), we get

$$\infty > \big(\sup_{n\in\mathbb{N}} |g(b)|^{n-1}\big) \sup_{\zeta\in\mathbb{C}} \frac{|g(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}} \ge \big(\sup_{n\in\mathbb{N}} |g(b)|^{n-1}\big) \frac{|g(b)|}{1+|b|}e^{-\frac{|b|^2}{2}},$$

which implies that $|g(b)| \le 1$. On the other hand, if $|g(b)| \le 1$, then

$$\sup_{n\in\mathbb{N}}\sup_{\zeta\in\mathbb{C}} \frac{|g(b)|^{n-1}|g(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}} \lesssim \sup_{n\in\mathbb{N}} |g(b)|^{n-1} < \infty$$

and hence, by (11), $J_{(g,\varphi)}$ is power bounded.

(b) First observe that

$$J^{n+1}_{(g,\varphi)} - J^n_{(g,\varphi)} = J_{(h_n,\varphi)}, \tag{12}$$

where $h_n(\zeta) = (g(b) - 1)(g(b))^{n-1}g(\zeta)$. Clearly, if $g(b) = 1$, $J_{(h_n,\varphi)}$ is a zero operator and the condition in (3) holds trivially.

(c) If $|g(b)| < 1$, then by (a) above, $J_{(g,\varphi)}$ is power bounded, and from (12) and (11), we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} n\|J^{n+1}_{(g,\varphi)} - J^n_{(g,\varphi)}\| &= \sup_{n\in\mathbb{N}} n\|J_{(h_n,\varphi)}\| \lesssim \sup_{n\in\mathbb{N}} n\big(\sup_{\zeta\in\mathbb{C}} \frac{|h_n(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}}\big)\\ &= \sup_{n\in\mathbb{N}} n\big(\sup_{\zeta\in\mathbb{C}} \frac{|g(b)-1||g(b)|^{n-1}|g(\zeta)|}{1+|\zeta|}e^{-\frac{|\zeta|^2}{2}}\big) \lesssim \sup_{n\in\mathbb{N}} n|g(b)|^{n-1} < \infty.\end{aligned}$$

Hence, $J_{(g,\varphi)}$ satisfies Ritt's resolvent condition. On the other hand, if $J_{(g,\varphi)}$ satisfies Ritt's resolvent condition, then a similar procedure as above gives

$$\begin{aligned}\infty > n\|J^{n+1}_{(g,\varphi)} - J^{n}_{(g,\varphi)}\| = n\|J_{(h_n,\varphi)}\| &\gtrsim n \sup_{\zeta\in\mathbb{C}} \frac{|g(b)-1||g(b)|^{n-1}|g(\zeta)|}{1+|\zeta|} e^{-\frac{|\zeta|^2}{2}} \\ &\geq n|g(b)|^{n-1}\Big(\frac{|g(b)-1||g(b)|}{1+|b|} e^{-\frac{|b|^2}{2}}\Big) \gtrsim n|g(b)|^n,\end{aligned}$$

from which the conclusion follows. □

4 Open Problem

In this paper, we studied power bounded and Ritt's resolvent growth condition for the generalized Volterra companion operator. A characterization of mean ergodic and uniformly mean ergodic properties of the operator remains open. Recall that, a continuous self map linear operator T on a Banach space $\mathcal{X}$ is said to be mean ergodic if there exists a continuous linear operator $\mathcal{P}$ on $\mathcal{X}$ such that

$$\mathcal{P}x := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k x,\ \ x \in \mathcal{X}$$

exists in $\mathcal{X}$. If the convergence is in the operator norm, then T is called uniformly mean ergodic.

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