

# Ideal Convergent Sequences Spaces Over n-normed Spaces

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## Abstract

*In the present paper we introduce some sequence spaces using ideal convergence and Musielak-Orlicz function  $M = (M_k)$  over n-normed spaces. We also examine some topological properties of the resulting sequence spaces.*

**Keywords:** *Paranorm, ideal convergence, Orlicz function, Musielak-Orlicz function.*

## 1 Introduction

The notion of ideal convergence was first introduced by Kostyrko [7] as a generalization of statistical convergence which was further studied in topological spaces by Das [1]. Gurdal investigated the relation between  $\mathcal{I}$ -cluster points and ordinary limit points of sequence in 2-normed space. More applications of ideals can be seen in [1] and [2]. We continue in this direction and introduce  $I$ -convergence of generalized sequences with respect to Musielak-Orlicz function.

A family  $\mathcal{I} \subset 2^X$  of subsets of a non empty set  $X$  is said to be an ideal in  $X$  if

1.  $\phi \in \mathcal{I}$ ,
2.  $A, B \in \mathcal{I}$  imply  $A \cup B \in \mathcal{I}$ ,
3.  $A \in \mathcal{I}$ ,  $B \subset A$  imply  $B \in \mathcal{I}$ ,

while an admissible ideal  $\mathcal{I}$  of  $X$  further satisfies  $\{x\} \in \mathcal{I}$  for each  $x \in X$  see [7]. A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is said to be I-convergent to  $x \in X$ , if for each  $\epsilon > 0$  the set  $A(\epsilon) = \left\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\right\}$  belongs to  $\mathcal{I}$  see in [7]. For more details about ideal convergence sequence spaces (see [8, 12, 14, 15, 16, 17, 18, 20, 24, 25]) and references therein.

The concept of 2-normed spaces was initially developed by Gähler [3] in the mid of 1960's, while that of  $n$ -normed spaces one can see in Misiak [11]. Since then, many others have studied this concept and obtained various results, see Gunawan ([4],[5]) and Gunawan and Mashadi [6]. Let  $n \in \mathbb{N}$  and  $X$  be a linear space over the field  $\mathbb{K}$ , where  $\mathbb{K}$  is field of real or complex numbers of dimension  $d$ , where  $d \geq n \geq 2$ . A real valued function  $\|\cdot, \dots, \cdot\|$  on  $X^n$  satisfying the following four conditions:

1.  $\|x_1, x_2, \dots, x_n\| = 0$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent in  $X$ ,
2.  $\|x_1, x_2, \dots, x_n\|$  is invariant under permutation,
3.  $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$  for any  $\alpha \in \mathbb{K}$ , and
4.  $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called a  $n$ -norm on  $X$ , and the pair  $(X, \|\cdot, \dots, \cdot\|)$  is called a  $n$ -normed space over the field  $\mathbb{K}$ . For example, we may take  $X = \mathbb{R}^n$  being equipped with the Euclidean  $n$ -norm  $\|x_1, x_2, \dots, x_n\|_E =$  the volume of the  $n$ -dimensional parallelopiped spanned by the vectors  $x_1, x_2, \dots, x_n$  which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$  for each  $i = 1, 2, \dots, n$ . Let  $(X, \|\cdot, \dots, \cdot\|)$  be a  $n$ -normed space of dimension  $d \geq n \geq 2$  and  $\{a_1, a_2, \dots, a_n\}$  be linearly independent set in  $X$ . Then the following function  $\|\cdot, \dots, \cdot\|_\infty$  on  $X^{n-1}$  defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an  $(n - 1)$ -norm on  $X$  with respect to  $\{a_1, a_2, \dots, a_n\}$ .

A sequence  $(x_k)$  in a  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is said to converge to some  $L \in X$  if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence  $(x_k)$  in a  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is said to be Cauchy if

$$\lim_{k, p \rightarrow \infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in  $X$  converges to some  $L \in X$ , then  $X$  is said to be complete with respect to the  $n$ -norm. Any complete  $n$ -normed space is said to be  $n$ -Banach space.

Let  $X$  be a linear metric space. A function  $p : X \rightarrow \mathbb{R}$  is called paranorm, if

1.  $p(x) \geq 0$ , for all  $x \in X$ ,
2.  $p(-x) = p(x)$ , for all  $x \in X$ ,
3.  $p(x + y) \leq p(x) + p(y)$ , for all  $x, y \in X$ ,
4. if  $(\lambda_n)$  is a sequence of scalars with  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$  and  $(x_n)$  is a sequence of vectors with  $p(x_n - x) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $p(\lambda_n x_n - \lambda x) \rightarrow 0$  as  $n \rightarrow \infty$ .

A paranorm  $p$  for which  $p(x) = 0$  implies  $x = 0$  is called total paranorm and the pair  $(X, p)$  is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [26], Theorem 10.4.2, P-183). For more details about sequence spaces, see [21, 22, 23] and references therein. The readers can refer to the recent monograph [27] and [28] on the sequence spaces and summability theory, and to the monograph [29] and the paper [30] on the Orlicz sequence spaces.

Let  $E$  be a sequence space. Then  $E$  is called solid (or normal) if  $(\alpha_k x_k) \in E$  whenever  $(x_k) \in E$  for all sequences  $(\alpha_k)$  of scalars with  $|\alpha_k| \leq 1$  for all  $k \in \mathbb{N}$ .  $E$  is called monotone provided that  $E$  contains the canonical preimages of all its step spaces. It is a well known result that if  $E$  is normal then it is monotone. A sequence space  $E$  is called sequence algebra if  $((x_k) \cdot (z_k)) = (x_k \cdot z_k) \in E$  whenever  $(x_k), (z_k) \in E$ .

An Orlicz function  $M : (0, \infty] \rightarrow (0, \infty]$  is a function, which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let  $w$  be the space of all real or complex sequences  $x = (x_k)$ , then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space  $\ell_M$  is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [9] that every Orlicz sequence space  $\ell_M$  contains a subspace isomorphic to  $\ell_p$  ( $p \geq 1$ ). An Orlicz function  $M$  is said to satisfy  $\Delta_2$ -condition

for all values  $x \geq 0$  if there exists a constant  $K > 0$  such that  $M(Lx) \leq KLM(x)$  for all values  $L > 1$ .

A sequence  $\mathcal{M} = (M_k)$  of Orlicz function is called a Musielak-Orlicz function see ([10, 19]). A sequence  $\mathcal{N} = (N_k)$  defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \quad k = 1, 2, \dots$$

is called the complementary function of a Musielak-Orlicz function  $\mathcal{M}$ . For a given Musielak-Orlicz function  $\mathcal{M}$ , the Musielak-Orlicz sequence space  $t_{\mathcal{M}}$  and its subspace  $h_{\mathcal{M}}$  are defined as follows

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \right\},$$

where  $I_{\mathcal{M}}$  is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider  $t_{\mathcal{M}}$  equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} \left( 1 + I_{\mathcal{M}}(kx) \right) : k > 0 \right\}.$$

Mursaleen and Noman [13] introduced the notion of  $\lambda$ -convergent and  $\lambda$ -bounded sequences as follows :

Let  $\lambda = (\lambda_k)_{k=1}^{\infty}$  be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \dots \quad \text{and} \quad \lambda_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty$$

and said that a sequence  $x = (x_k) \in w$  is  $\lambda$ -convergent to the number  $L$ , called the  $\lambda$ -limit of  $x$  if  $\Lambda_m(x) \rightarrow L$  as  $m \rightarrow \infty$ , where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1})x_k.$$

The sequence  $x = (x_k) \in w$  is  $\lambda$ -bounded if  $\sup_m |\Lambda_m(x)| < \infty$ . It is well known [13] that if  $\lim_m x_m = a$  in the ordinary sense of convergence, then

$$\lim_m \left( \frac{1}{\lambda_m} \left( \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that  $\lim_m \Lambda_m(x) = a$  and hence  $x = (x_k) \in w$  is  $\lambda$ -convergent to  $a$ .

Let  $\mathcal{M} = (M_k)$  be a Musielak-Orlicz function,  $p = (p_k)$  be a bounded sequence of positive real numbers. We define the following sequence spaces over  $n$ -normed spaces in the present paper:

$$c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \right. \\ \left. \text{for some } L \text{ and } \rho > 0 \right\},$$

$$c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \right. \\ \left. \text{for some } \rho > 0 \right\},$$

and

$$l_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. \sup_k \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \right. \\ \left. \text{for some } \rho > 0 \right\},$$

We can write

$$m^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) = c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \cap l_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$$

and

$$m_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) = c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \cap l_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|).$$

If we take  $p_k = 1$  for all  $k \in \mathbb{N}$ , we have

$$c^I(\mathcal{M}, \Lambda, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. I - \lim_k M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) = 0, \right. \\ \left. \text{for some } L \text{ and } \rho > 0 \right\},$$

$$c_0^I(\mathcal{M}, \Lambda, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. I - \lim_k M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) = 0, \right. \\ \left. \text{for some } \rho > 0 \right\},$$

and

$$l_\infty(\mathcal{M}, \Lambda, \|\cdot, \dots, \cdot\|) = \left\{ x = (x_k) \in w : \right. \\ \left. \sup_k M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) < \infty, \right. \\ \left. \text{for some } \rho > 0 \right\},$$

The following inequality will be used throughout the paper. If  $0 \leq p_k \leq \sup p_k = H$ ,  $D = \max(1, 2^{H-1})$  then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (1)$$

for all  $k$  and  $a_k, b_k \in \mathbb{C}$ . Also  $|a|^{p_k} \leq \max(1, |a|^H)$  for all  $a \in \mathbb{C}$ .

The main aim of this paper is to study some ideal convergence sequence spaces defined by a Musielak-Orlicz function  $\mathcal{M} = (M_k)$  over  $n$ -normed spaces. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

## 2 Main Results

**Theorem 2.1** *Let  $M = (M_k)$  be a Musielak-Orlicz function,  $p = (p_k)$  be a bounded sequence of positive real numbers. Then the spaces  $c^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$ ,  $c_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$ ,  $m^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and  $m_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  are linear over the complex field  $\mathbb{C}$ .*

**Proof.** Let  $x, y \in c^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and let  $\alpha, \beta$  be scalars. Then there exist positive numbers  $\rho_1$  and  $\rho_2$  such that

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L_1}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \text{ for some } L_1 \in \mathbb{C}$$

and

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(y) - L_2}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \text{ for some } L_2 \in \mathbb{C}.$$

For a given  $\epsilon > 0$ , we have

$$D_1 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L_1}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right\} \quad (2)$$

and

$$D_2 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \left\| \frac{\Lambda_k(y) - L_2}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right\}. \quad (3)$$

Let  $\rho_3 = \max \{2|\alpha|\rho_1, 2|\beta|\rho_2\}$ . Since  $\mathcal{M} = (M_k)$  is non-decreasing, convex function and so by using inequality (1), we have

$$\begin{aligned} & \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k((\alpha x + \beta y) - (\alpha L_1 + \beta L_2))}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \lim_k \left[ M_k \left( \left\| \frac{\alpha[\Lambda_k(x) - L_1]}{\rho_3}, z_1, \dots, z_{n-1} \right\| + \left\| \frac{\beta[\Lambda_k(y) - L_2]}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L_1}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(y) - L_2}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned}$$

Now by (2) and (3), we have

$$\left\{ k \in \mathbb{N} : \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k((\alpha x + \beta y) - (\alpha L_1 + \beta L_2))}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \epsilon \right\} \subset D_1 \cup D_2.$$

Therefore  $\alpha x + \beta y \in c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Hence  $c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  is a linear space. Similarly we can prove that  $c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ ,  $m^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and  $m_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  are linear spaces. ■

**Theorem 2.2** *Let  $M = (M_k)$  be a Musielak-Orlicz function. Then*

$$c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \subset c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \subset \ell_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|).$$

**Proof.** Let  $x \in c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Then there exist  $L \in \mathbb{C}$  and  $\rho > 0$  such that

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0.$$

We have

$$\begin{aligned} \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} & \leq \frac{1}{2} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + \left[ M_k \frac{1}{2} \left( \left\| \frac{L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned}$$

Taking supremum over  $k$  on both sides, we get  $x \in \ell_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . The inclusion  $c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \subset c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  is obvious. Thus

$$c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \subset c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|) \subset \ell_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|).$$

This completes the proof of the theorem. ■

**Theorem 2.3** *Let  $M = (M_k)$  be a Musielak-Orlicz function,  $p = (p_k)$  be a bounded sequence of positive real numbers. Then  $l_\infty(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  is a paranormed space equipped with the paranorm defined by*

$$g(x) = \inf \left\{ \rho > 0 : \sup_k M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \leq 1 \right\}.$$

**Proof.** It is clear that  $g(x) = g(-x)$ . Since  $M_k(0) = 0$ , we get  $g(0) = 0$ . Let us take  $x, y \in l_\infty(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Let

$$B(x) = \left\{ \rho > 0 : \sup_k \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \right\},$$

$$B(y) = \left\{ \rho > 0 : \sup_k \left[ M_k \left( \left\| \frac{\Lambda_k(y)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1, \right\}.$$

Let  $\rho_1 \in B(x)$  and  $\rho_2 \in B(y)$ . If  $\rho = \rho_1 + \rho_2$ , then we have

$$\begin{aligned} \sup_k M_k \left( \left\| \frac{\Lambda_k(x+y)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) & \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_k M_k \left( \left\| \frac{\Lambda_k(x)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \\ & \quad + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_k M_k \left( \left\| \frac{\Lambda_k(y)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right). \end{aligned}$$

Thus  $\sup_k M_k \left( \left\| \frac{\Lambda_k(x+y)}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \leq 1$  and

$$\begin{aligned} g(x+y) & \leq \inf \left\{ (\rho_1 + \rho_2) > 0 : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \\ & \leq \inf \left\{ \rho_1 > 0 : \rho_1 \in B(x) \right\} + \inf \left\{ \rho_2 > 0 : \rho_2 \in B(y) \right\} \\ & = g(x) + g(y). \end{aligned}$$

Let  $\sigma^s \rightarrow \sigma$  where  $\sigma, \sigma^s \in \mathbb{C}$  and let  $g(x^s - x) \rightarrow 0$  as  $s \rightarrow \infty$ . We have to show that  $g(\sigma^s x^s - \sigma x) \rightarrow 0$  as  $s \rightarrow \infty$ . Let

$$B(x^s) = \left\{ \rho_s > 0 : \sup_k M_k \left( \left\| \frac{\Lambda_k(x^s)}{\rho_s}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \leq 1, \right\},$$

$$B(x^s - x) = \left\{ \rho'_s > 0 : \sup_k M_k \left( \left\| \frac{\Lambda_k(x^s - x)}{\rho'_s}, z_1, \dots, z_{n-1} \right\| \right)^{p_k} \leq 1, \right\}.$$

If  $\rho_s \in B(x^s)$  and  $\rho'_s \in B(x^s - x)$  then we observe that

$$\begin{aligned} & M_k \left( \left\| \frac{\Lambda_k(\sigma^s x^s - \sigma x)}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \right) \\ & \leq M_k \left( \left\| \frac{\Lambda_k(\sigma^s x^s - \sigma x^s)}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} + \frac{|(\sigma x^s - \sigma x)|}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \right) \\ & \leq \frac{|\sigma^s - \sigma| \rho_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} M_k \left( \left\| \frac{\Lambda_k(x^s)}{\rho_s}, z_1, \dots, z_{n-1} \right\| \right) \\ & \quad + \frac{|\sigma| \rho'_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} M_k \left( \left\| \frac{\Lambda_k(x^s - x)}{\rho'_s}, z_1, \dots, z_{n-1} \right\| \right). \end{aligned}$$

From the above inequality, it follows that

$$\left[ M_k \left( \left\| \frac{\Lambda_k(\sigma^s x^s - \sigma x)}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1$$

and consequently,

$$\begin{aligned} g(\sigma^s x^s - \sigma x) & \leq \inf \left\{ \left( \rho_s |\sigma^s - \sigma| + \rho'_s |\sigma| \right) > 0 : \rho_s \in B(x^s), \rho'_s \in B(x^s - x) \right\} \\ & \leq (|\sigma^s - \sigma|) > 0 \inf \left\{ \rho > 0 : \rho_s \in B(x^s) \right\} \\ & \quad + (|\sigma|) > 0 \inf \left\{ (\rho'_s)^{\frac{p_n}{H}} : \rho'_s \in B(x^s - x) \right\} \\ & \rightarrow 0 \text{ as } s \rightarrow \infty. \end{aligned}$$

This completes the proof. ■

**Theorem 2.4** Let  $M' = (M'_k)$  and  $M'' = (M''_k)$  are Musielak-Orlicz functions that satisfies the  $\Delta_2$ -condition. Then

(i)  $Z(M'', \Lambda, p, \|\cdot, \dots, \cdot\|) \subseteq Z(M' \circ M'', \Lambda, p, \|\cdot, \dots, \cdot\|)$ ,

(ii)  $Z(M', \Lambda, p, \|\cdot, \dots, \cdot\|) \cap Z(M'', \Lambda, p, \|\cdot, \dots, \cdot\|) \subseteq Z(M' + M'', \Lambda, p, \|\cdot, \dots, \cdot\|)$   
for  $Z = c^I, c_0^I, m^I, m_0^I$ .

**Proof.** Let  $x \in c_0^I(\mathcal{M}'', \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Then there exists  $\rho > 0$  such that

$$I - \lim_k \left[ M''_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \quad (4)$$

Let  $\epsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$  such that  $M'_k(t) < \epsilon$  for  $0 \leq t \leq \delta$ .

Write  $y_k = \left[ M''_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}$  and consider

$$\lim_{\substack{0 \leq y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) = \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) + \lim_{\substack{y_k > \delta \\ k \in \mathbb{N}}} M'_k(y_k).$$

Since  $\mathcal{M} = (M_k)$  satisfies  $\Delta_2$ -condition, we have

$$\lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) \leq M'_k(2) \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} (y_k). \quad (5)$$

For  $y_k > \delta$ , we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since  $\mathcal{M}' = (M'_k)$  is non-decreasing and convex, it follows that

$$M'_k(y_k) < M'_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}M'_k(2) + \frac{1}{2}\frac{M'_k(2y_k)}{\delta}.$$

Since  $\mathcal{M}' = (M'_k)$  is satisfies  $\Delta_2$ -condition, we have

$$M'_k(y_k) < \frac{1}{2}K\frac{y_k}{\delta}M'_k(2) + \frac{1}{2}K\frac{y_k}{\delta}M'_k(2) = K\frac{y_k}{\delta}M'_k(2).$$

Hence

$$\lim_{\substack{y_k > \delta \\ k \in \mathbb{N}}} M'_k(y_k) \leq \max(1, K\delta^{-1}M'_k(2)) \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} (y_k) \quad (6)$$

From (4), (5) and (6), we have  $x = (x_k) \in c_0^I(\mathcal{M}' \circ \mathcal{M}'', \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Thus  $c_0^I(\mathcal{M}'', \Lambda, p, \|\cdot, \dots, \cdot\|) \subseteq c_0^I(\mathcal{M}' \circ \mathcal{M}'', \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Similarly we can prove the other cases.

(ii) Let  $x \in c_0^I(\mathcal{M}', \Lambda, p, \|\cdot, \dots, \cdot\|) \cap c_0^I(\mathcal{M}'', \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Then there exists  $\rho > 0$  such that

$$I - \lim_k \left[ M'_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0$$

and

$$I - \lim_k \left[ M''_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0.$$

The rest of the proof follows from the following equality

$$\begin{aligned} & \lim_k \left[ (M'_k + M''_k) \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &= \lim_{k \in \mathbb{N}} \left[ M'_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} + \lim_{k \in \mathbb{N}} \left[ M''_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned}$$

■

**Corollary 2.5** *Let  $M = (M_k)$  be a Musielak -Orlicz function which satisfies  $\Delta_2$ -condition. Then  $Z(p, \Lambda, \|\cdot, \dots, \cdot\|) \subseteq Z(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  for  $Z = c^I, c_0^I, m^I$  and  $m_0^I$ .*

**Proof.** The proof follows from Theorem 2.3 by putting  $M_k''(x) = x$  and  $M_k'(x) = M_k(x) \forall x \in [0, \infty)$ . ■

**Theorem 2.6** *The spaces  $c_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and  $m_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  are solid.*

**Proof.** We give the proof for the space  $c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Let  $x \in c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Then there exists  $\rho > 0$  such that

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \quad (7)$$

Let  $(\alpha_k)$  be a sequence of scalars with  $|\alpha_k| \leq 1 \quad \forall k \in \mathbb{N}$ . Then, the result follows from the following inequality

$$\lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(\alpha x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \leq \lim_k M_k \left( \left\| \frac{\Lambda_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right)^{p_k}$$

and this completes the proof. Similarly, we can give the proof for the space  $m_0^I(\mathcal{M}, \Lambda, p)$ . ■

**Corollary 2.7** *The spaces  $c_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and  $m_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  are monotone.*

**Proof.** This is easy. So, we omit the details. ■

**Theorem 2.8** *The spaces  $c^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  and  $c_0^I(M, \Lambda, p, \|\cdot, \dots, \cdot\|)$  are sequence algebra.*

**Proof.** Let  $x, y \in c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Then

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \text{ for some } \rho_1 > 0$$

and

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(y)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0, \text{ for some } \rho_2 > 0.$$

Let  $\rho = \rho_1 + \rho_2$ . Then we can show that

$$I - \lim_k \left[ M_k \left( \left\| \frac{\Lambda_k(x \cdot y)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0.$$

Thus  $(x \cdot y) \in c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$ . Hence  $c_0^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  is also a sequence algebra. Similarly, we can prove that  $c^I(\mathcal{M}, \Lambda, p, \|\cdot, \dots, \cdot\|)$  is a sequence algebra. ■

### 3 Open Problem

We introduced some sequence spaces using ideal convergence and Musielak-Orlicz function  $M = (M_k)$  over  $n$ -normed spaces. For the reference sections, consider the following introduction described the main results are motivating the research. Thus it will be possible to benefit from this article for new studies.

### References

- [1] P. Das, P. Kostyrko, W. Wilczynski and P. Malik, I and  $I^*$  convergence of double sequences, *Math. Slovaca*, **58** (2008), 605–620.
- [2] P. Das and P. Malik, On the statistical and I-variation of double sequences, *Real Anal. Exchange*, **33(2)** (2007-2008), 351–364.
- [3] S. Gähler, Linear 2-normierte Räume, *Math. Nachr.*, **28** (1965), 1–43.
- [4] H. Gunawan, On  $n$ -Inner Product,  $n$ -Norms, and the Cauchy-Schwartz Inequality, *Sci. Math. Jap.*, **5** (2001), 47–54.
- [5] H. Gunawan, The space of  $p$ -summable sequence and its natural  $n$ -norm, *Bull. Aust. Math. Soc.*, **64** (2001), 137–147.
- [6] H. Gunawan and M. Mashadi, On  $n$ -normed spaces, *Int. J. Math. Math. Sci.*, **27** (2001), 631–639.
- [7] P. Kostyrko, T. Salat and W. Wilczynski, I-Convergence, *Real Anal. Exchange*, **26(2)** (2000), 669–686.
- [8] V. Kumar, On I and  $I^*$  convergence of double sequences, *Math. Commun.*, **12** (2007), 171–181.
- [9] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.* **10** (1971), 345–355.
- [10] L. Maligranda, Orlicz spaces and interpolation, *Seminars in Mathematics 5*, Polish Academy of Science, 1989.
- [11] A. Misiak,  $n$ -inner product spaces, *Math. Nachr.*, **140** (1989), 299–319.
- [12] M. Mursaleen and A. Alotaibi, On I-convergence in random 2-normed spaces, *Math. Slovaca*, **61(6)** (2011), 933–940.
- [13] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces  $l_p$  and  $l_\infty$ , *Filomat*, **25** (2011), 33–51.

- [14] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces  $l_p$  and  $l_\infty$  II, *Math. Commun.*, **16** (2011), 383–398.
- [15] M. Mursaleen, S. A. Mohiuddine and O. H. H. Edely, On ideal convergence of double sequences in intuitionistic fuzzy normed spaces, *Comput. Math. Appl.*, **59** (2010), 603–611.
- [16] M. Mursaleen and S. A. Mohiuddine, On ideal convergence of double sequences in probabilistic normed spaces, *Math. Reports*, **12(64)(4)** (2010), 359–371.
- [17] M. Mursaleen and S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, *Math. Slovaca*, **62** (2012), 49–62.
- [18] M. Mursaleen and S. K. Sharma, Spaces of ideal convergent sequences, *The World Scientific Journal*, Vol. 2014, 6 pages.
- [19] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034 (1983).
- [20] K. Raj and S. K. Sharma, Ideal convergent sequence spaces defined by a Musielak-Orlicz function, *Thai J. Math.*, **11** (2013), 577–587.
- [21] K. Raj and S. K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz functions, *Acta Univ. Sapientiae Math.*, **3** (2011), 97–109.
- [22] K. Raj and S. K. Sharma, Some generalized difference double sequence spaces defined by a sequence of Orlicz-function, *Cubo*, **14** (2012), 167–189.
- [23] K. Raj and S. K. Sharma, Some multiplier sequence spaces defined by a Musielak-Orlicz function in  $n$ -normed spaces, *New Zealand J. Math.*, **42** (2012), 45–56.
- [24] A. Şahiner, M. Gürdal, S. Saltan and H. Gunawan, On ideal convergence in 2-normed spaces, *Taiwanese J. Math.*, **11** (2007), 1477–1484.
- [25] B. C. Tripathy and B. Hazarika, Some  $I$ -convergent sequence spaces defined by Orlicz functions, *Acta Mathematicae Applicatae Sinica*, **27** (2011), 149–154.
- [26] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stud. 85 (1984).
- [27] F. Başar, *Summability Theory and its Applications*, Bentham Science Publishers, İstanbul, 2012.

- [28] M. Mursaleen, F. Başar, *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Taylor and Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.
- [29] F. Başar, H. Dutta, *Summable Spaces and Their Duals, Matrix Transformations and Geometric Properties*, CRC Press, Taylor and Francis Group, Monographs and Research Notes in Mathematics, Boca Raton, London, New York, 2020.
- [30] H. Dutta, F. Başar, A generalization of Orlicz sequence spaces by Cesáro mean of order one, *Acta Math. Univ. Comen.*, **80(2)** (2011), 185–200.
- [31] M. Gurdal, On Ideal Convergent Sequences in 2-Normed Spaces, *Thai Journal of Mathematics*, **4(1)** (2006), 85–91.