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Coefficient Estimates of a New Bi-Univalent Function Class Introduced by Lucas-Balancing Polynomial

Arzu Akgül

Department of Mathematics,Kocaeli University,410001 Kocaeli
Turkey
e-mail: akgul@kocaeli.edu.tr

Abstract

In this study, we introduce and investigate a new class $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}(\zeta,\delta,n;\mathcal{R}(x,z))$ ***of bi-univalent analytic functions via Al-Oboudi differential operator, Lucas-Balancing polynomial and principle of subordination in open unit disc. Also we examine the estimates on the coefficients*** $|a_2|$ ***and*** $|a_3|$ ***for functions in*** $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}(\zeta,\delta,n;\mathcal{R}(x,z))$

Keywords: *Holomorphic functions, Univalent functions, Bi-Univalent functions, Coefficient estimates.*

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1 Introduction

Let A indicate the class of analytic functions of the form

$$\mathfrak{K}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $E = \{z : |z| < 1\}$,and let $S = \mathfrak{K} \in A : \mathfrak{K}$is univalent in$E$.

According to the Koebe one-quarter theorem [10] , the range of every function $\mathfrak{K} \ \in S$ contains the disc of radius $\left\{w : |w| < \frac{1}{4}\right\}$. Hence every such function $\mathfrak{K} \in S$ has a satisfied inverse function $\mathfrak{K}^{-1}$

$$\mathfrak{K}^{-1}(\mathfrak{K}(z)) = z, (z \in E)$$

and

$$\mathfrak{K}\left(\mathfrak{K}^{-1}(w)\right) = w\left(|w| < r_0(\mathfrak{K}), r_0(\mathfrak{K}) \geq \frac{1}{4}\right),$$

where

$$\mathfrak{K}^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

If $\mathfrak{K}$ and $\mathfrak{K}^{-1}$ are univalent in E, then $\mathfrak{K} \in A$ is said to be bi-univalent in E. The class of bi-univalent functions defined on the unit disk E is shown by Σ. It is well known that the calss Σ is not empty. We can give some examples for functions and their invers functions belonging the class Σ :

$$\begin{aligned}
\mathfrak{K}_1(z) &= \frac{z}{z-1}, \mathfrak{K}_1^{-1}(w) = \frac{w}{w+1}; \\
\mathfrak{K}_2(z) &= \frac{1}{2}\log\frac{1+z}{1-z}, \mathfrak{K}_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}; \\
\mathfrak{K}_3(z) &= -\log(1-z), \mathfrak{K}_3^{-1}(w) = \frac{e^{w}-1}{e^{w}}.
\end{aligned}$$

For analytic functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$, $\mathfrak{K}_1$ is named to be subordinate to $\mathfrak{K}_2$, indicated

$$\mathfrak{K}_1(z) \prec \mathfrak{K}_2(z), \tag{3}$$

if there is an analytic function $\mathcal{W}$ such that

$$\mathcal{W}(0) = 0 \ , \ |\mathcal{W}(z)| < 1 \ \text{ and } \mathfrak{K}_1(z) = \mathfrak{K}_2(\mathcal{W}(z))$$

For a function $\mathfrak{K} \in A$, Al-Oboudi [7] defined the following differantial operator, named Al-Oboudi differantial operator:

$$\begin{aligned}
\mathcal{D}_\delta^0 \mathfrak{K}(z) &= \mathfrak{K}(z) && (4)\\
\mathcal{D}_\delta^1 \mathfrak{K}(z) &= (1-\delta)\mathfrak{K}(z) + \delta z \mathfrak{K}'(z) = \mathcal{D}_\delta \mathfrak{K}(z), \delta \geq 0, && (5)\\
\mathcal{D}_\delta^n \mathfrak{K}(z) &= \mathcal{D}_\delta\left(\mathcal{D}_\delta^{n-1}\mathfrak{K}(z)\right), n \in \mathbb{N} = \{1, 2, ...\}. && (6)
\end{aligned}$$

If $\mathfrak{K}$ is explained by (1), then by means of (4) and (5) we see that,

$$\begin{aligned}
\mathcal{D}_\delta^n \mathfrak{K}(z) &= z + \sum_{k=2} [1 + (k-1)\delta]^n a_k z^k, && (7)\\
n &\in \mathbb{N}_0 = \{0, 1, 2, ...\}.
\end{aligned}$$

with $\mathcal{D}_\delta^n \mathfrak{K}(0) = 0$.When $\delta = 1$, then we get Sălăgean's differantial operator [19].

The concept of Balancing number was defined by Behera and Panda in [8]. Actually, balancing number n and its balancer r are solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

If n is a balancing number, then $8n^2+1$ is a perfect square also its positive square root is named a Lucas-Balancing number [18]. Nowadays, these new number sequences have been studied by some researchers and its some generalizations have been defined. One can find comprehensive knowledge about Lucas-Balancing numbers in [9, 11, 12, 13, 14] and references therein. Lucas-Balancing polynomial is a natural extension of Lucas-Balancing numbers and defined by following:

Definition 1.1 *[16] Let $x \in \mathbb{C}$ and $n \geq 2$. Then, Lucas-Balancing polynomials are defined with the following recurrence relation*

$$\mathfrak{C}_n(x) = 6x\mathfrak{C}_{n-1}(x) - \mathfrak{C}_{n-2}(x), \tag{8}$$

where

$$\begin{aligned} \mathfrak{C}_0(x) &= 1, \\ \mathfrak{C}_1(x) &= 3x. \end{aligned} \tag{9}$$

By using recurrence relation given by (8) it s easily obtained that

$$\begin{aligned} \mathfrak{C}_2(x) &= 18x^2 - 1, \\ \mathfrak{C}_3(x) &= 108x^3 - 9x. \end{aligned} \tag{10}$$

Lemma 1.2 *[16]The ordinary generating function of the Lucas-Balancing polynomials is explained by*

$$R(x,z) = \sum_{n=0}^{\infty} \mathfrak{C}_n(x) z^n = \frac{1-3xz}{1-6xz+z^2}, \tag{11}$$

where $x \in \mathbb{C}$ and $n \geq 2, z \in \mathrm{E}$.

In the last decades, due to the fact that they can applicable to number theory, numerical analysis, combinatorics, and other fields, theory and applications of Fibonacci, Lucas, Chebyshev, Lucas-Lehmer, Lucas-Balancing polynomials, in modern science have gained very importance. Nowadays, these kind of polyinomials have been investigated by many authors see [1, 2, 3, 4, 5, 6, 17, 20, 21, 22].

2 The class $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$

In this section we define the new class $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$ via Al-Oboudi differential operator, Lucas-Balancing polynomial and principle of subordination. Also we examine the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$.

Definition 2.1 *We say that* $\mathfrak{K}$ *of the form (1) is in the class* $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$, *if the following subordinations hold:*

$$(1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)}{z}+\zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)\right)'\prec\mathcal{R}(x,z) \tag{12}$$

and

$$(1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{g}(w)}{w}+\zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{g}(w)\right)'\prec\mathcal{R}(x,w) \tag{13}$$

where $\zeta\geq 1,\delta\geq 0, x\in\mathbb{C}\backslash\left\{\mp\frac{1}{3}\frac{(1+\zeta)(1+\delta)^{n}}{\sqrt{2(1+\zeta)^{2}(1+\delta)^{2n}-(1+2\zeta)(1+2\delta)^{n}}}\right\}$, $z,w\in\mathrm{E}$, *the function* $\mathcal{D}_{\delta}^{n}$ *is Al Oboudi differential operator and* $\mathfrak{g}$ *is given by (2) and* $\mathcal{R}(x,z)$ *is given by (11).*

Throughout this study, the functions $\mathfrak{K}$ and $\mathfrak{g}=\mathfrak{K}^{-1}$ are given by (1) and (2) and complex numbers $z,w\in E$.

When allocating the parameters ζ,δ and n, we can obtain some new subclasses of Σ,as indicated in the following examples:

Example 2.2 *We say that* $\mathfrak{K}$ *is in the class*

$$\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,1,n;\mathcal{R}\left(x,z\right)\right)=\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,n;\mathcal{R}\left(x,z\right)\right),$$

if the following subordinations hold:

$$(1-\zeta)\frac{\mathcal{D}^{n}\mathfrak{K}(z)}{z}+\zeta\left(\mathcal{D}^{n}\mathfrak{K}(z)\right)'\prec\mathcal{R}\left(x,z\right)$$

and

$$(1-\zeta)\frac{\mathcal{D}^{n}\mathfrak{g}(w)}{w}+\zeta\left(\mathcal{D}^{n}\mathfrak{g}(w)\right)'\prec\mathcal{R}\left(x,w\right)$$

where $\mathcal{D}^{n}$ *is the Sălăgean differential operator and* $x\in\mathbb{C}\backslash\left\{\mp\frac{2^{n}}{3}\frac{(1+\zeta)}{\sqrt{2^{2n+1}(1+\zeta)^{2}-3^{n}(1+2\zeta)}}\right\}$.

In this example, besides $\delta=1$ *by choosing* $\zeta=1$ *and* $n=0$, *we have a new subclass :*

We say that $\mathfrak{K}$ *is in the class*

$$\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(1,1,0;\mathcal{R}\left(x,z\right)\right)=\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\mathcal{R}\left(x,z\right)\right),$$

if the following subordinations hold

$$\mathfrak{K}'(z) \prec \mathcal{R}\left(x,z\right)$$

and

$$\mathfrak{g}'(w) \prec \mathcal{R}\left(x,w\right),$$

where $x \in \mathbb{C}\backslash \left\{ \mp \frac{2\sqrt{5}}{15} \right\}$.

Example 2.3 *We say that* $\mathfrak{K}$ *is in the class*

$$\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,0;\mathcal{R}\left(x,z\right)\right) = \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta;\mathcal{R}\left(x,z\right)\right)$$

if the following subordinations hold:

$$(1-\zeta)\frac{\mathfrak{K}(z)}{z} + \zeta\mathfrak{K}'(z) \prec \mathcal{R}(x,z)$$

and

$$(1-\zeta)\frac{\mathcal{D}^n\mathfrak{g}(w)}{w} + \zeta\left(\mathcal{D}^n\mathfrak{g}(w)\right)' \prec \mathcal{R}(x,w),$$

where $x \in \mathbb{C}\backslash \left\{ \mp \frac{1}{3}\frac{(1+\zeta)}{\sqrt{(1+\zeta)^2+\zeta^2}} \right\}$

Example 2.4 *We say that* $\mathfrak{K}$ *is in the class*

$$\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(1,\delta,n;\mathcal{R}\left(x,z\right)\right) = \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\delta,n;\mathcal{R}\left(x,z\right)\right)$$

if the following subordinations hold:

$$(1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)}{z} + \zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)\right)' \prec \mathcal{R}\left(x,z\right)$$

and

$$(1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{g}(w)}{w} + \zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{g}(w)\right)' \prec \mathcal{R}\left(x,w\right), \tag{14}$$

where $x \in \mathbb{C}\backslash \left\{ \mp \frac{2}{3}\frac{(1+\delta)^n}{\sqrt{8(1+\delta)^{2n}-3(1+2\delta)^n}} \right\}$.

Theorem 2.5 *Let* $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$. *Then,*

$$|a_2| \leq \frac{3\left|x\right|\sqrt{3x}}{\sqrt{9x^2(1+2\zeta)(1+2\delta)^n - \left(18x^2-1\right)(1+\zeta)^2(1+\delta)^{2n}}}. \tag{15}$$

and

$$|a_3| \leq 3\left|x\right|\left[\frac{3\left|x\right|}{(1+\delta)^{2n}\left(1+\zeta\right)^2} + \frac{1}{(1+2\delta)^n\left(1+2\zeta\right)}\right]. \tag{16}$$

where $\zeta \geq 1, \delta \geq 0, x \in \mathbb{C}\backslash\mp\left\{ \frac{(1+\zeta)(1+\delta)^n}{3}\frac{1}{\sqrt{(1+\zeta)^2(1+\delta)^{2n}-(1+2\zeta)(1+2\delta)^n}} \right\}, z, w \in$ E.

Let $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}(\zeta, \delta, n; \mathcal{R}(x, z))$. We have the following from the definition in formulas (12) and (13)

$$(1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)}{z} + \zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)\right)' = \mathcal{R}\left(x, phi(z)\right) \tag{17}$$

and

$$(1-\zeta)\frac{\mathcal{D}^{n}\mathfrak{g}(w)}{w} + \zeta\left(\mathcal{D}^{n}\mathfrak{g}(w)\right)' = \mathcal{R}\left(x, \psi(w)\right) \tag{18}$$

where the analytical functions ϕ and ψ have the form

$$\begin{aligned} \phi(z) &= \phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \cdots, \qquad (19) \\ \psi(w) &= \psi_1 w + \psi_2 w^2 + \psi_3 w^3 + \cdots. \qquad (20) \end{aligned}$$

and $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1, |\psi(w)| < 1$ .It is known that if for $z, w \in E$,

$$|\phi(z)| = \left|\sum_{i=1}^{n} \phi_i z^i\right| < 1$$

and

$$|\psi(w)| = \left|\sum_{i=1}^{n} \psi_i w^i\right| < 1.$$

Thus

$$|\phi_i| < 1 \tag{21}$$

and

$$|\psi_i| < 1 \tag{22}$$

where $i \in \mathbb{N}$. Now, after putting (19) and (20) in (17) and (18), by a simple calculation, we see that

$$\begin{aligned} (1-\zeta)\frac{\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)}{z} + \zeta\left(\mathcal{D}_{\delta}^{n}\mathfrak{K}(z)\right)' &= \mathfrak{C}_0(x) + \mathfrak{C}_1(x)\phi_1 z + \left[\mathfrak{C}_1(x)\phi_2 + \mathfrak{C}_2(x)\phi_1^2\right] z^2 \\ &\quad + \left[\mathfrak{C}_1(x)\phi_3 + 2\mathfrak{C}_2(x)\phi_1\phi_2 + \mathfrak{C}_3(x)\phi_1^3\right] z^3 + (23) \end{aligned}$$

and

$$\begin{aligned} (1-\zeta)\frac{\mathcal{D}_{\delta}^{n}v(w)}{w} + \zeta\left(\mathcal{D}_{\delta}^{n}v(w)\right)' &= \mathfrak{C}_0(x) + \mathfrak{C}_1(x)\psi_1 w + \left[\mathfrak{C}_1(x)\psi_2 + \mathfrak{C}_2(x)\psi_1^2\right] w^2 \\ &\quad + \left[\mathfrak{C}_1(x)\psi_3 + 2\mathfrak{C}_2(x)\psi_1\psi_2 + \mathfrak{C}_3(x)\psi_1^3\right] w^3 + (24) \end{aligned}$$

So, comparing the corresponding coefficients in (23)and (24), we get

$$\begin{aligned}(1+\zeta)(1+\delta)^n a_2 &= \mathfrak{C}_1(x)\phi_1 &(25)\\ (1+2\zeta)(1+2\delta)^n a_3 &= \mathfrak{C}_1(x)\phi_2+\mathfrak{C}_2(x)\phi_1^2, &(26)\\ -(1+\zeta)(1+\delta)^n a_2 &= \mathfrak{C}_1(x)\psi_1 &(27)\\ (1+2\zeta)(1+2\delta)^n(2a_2^2-a_3) &= \mathfrak{C}_1(x)\psi_2+\mathfrak{C}_2(x)\psi_1^2. &(28)\end{aligned}$$

From (25)and (27)

$$\phi_1=-\psi_1 \tag{29}$$

and

$$2(1+\zeta)^2(1+\delta)^{2n}a_2^2=\mathfrak{C}_1^2(x)\left(\phi_1^2+\psi_1^2\right). \tag{30}$$

Adding (26)and (28) we get

$$2(1+2\zeta)(1+2\delta)^n a_2^2=\mathfrak{C}_1(x)\left(\phi_2+\psi_2\right)+\mathfrak{C}_2(x)\left(\phi_1^2+\psi_1^2\right) \tag{31}$$

By using (30) in (31) we have

$$a_2^2=\frac{\mathfrak{C}_1^3(x)(x)\left(\phi_2+\psi_2\right)}{2\left[\mathfrak{C}_1^2(x)(1+2\zeta)(1+2\delta)^n-\mathfrak{C}_2(x)(1+\zeta)^2(1+\delta)^{2n}\right]} \tag{32}$$

Concidering equalities (9) and (10) and using them in (32), we get

$$a_2^2=\frac{27x^3\left(\phi_2+\psi_2\right)}{2\left[9x^2(1+2\zeta)(1+2\delta)^n-\left(18x^2-1\right)(1+\zeta)^2(1+\delta)^{2n}\right]}. \tag{33}$$

After taking square root and taking mudule both sides of the inequality (33), also using (21) and (22), we obtain desired result given in (15).

To obtain (16), firstly, subtructing (28) from (26) and using (29), then we get

$$2(1+2\zeta)(1+2\delta)^n\left(a_3-a_2^2\right)=\mathfrak{C}_1(x)\left(\phi_2-\psi_2\right). \tag{34}$$

Also, by using (30)in (34) ,we have

$$a_3=\frac{\mathfrak{C}_1^2(x)\left(\phi_1^2+\psi_1^2\right)}{2(1+\zeta)^2(1+\delta)^{2n}}+\frac{\mathfrak{C}_1(x)\left(\phi_2-\psi_2\right)}{2(1+2\zeta)(1+2\delta)^n}. \tag{35}$$

Taking module of the both side of (35) also by the help of (21), (22), (29),

$$|a_3|\leq\frac{\mathfrak{C}_1^2(x)}{(1+\delta)^{2n}\left(1+\zeta\right)^2}+\frac{|\mathfrak{C}_1(x)|}{(1+2\delta)^n\left(1+2\zeta\right)}. \tag{36}$$

By writing $\mathfrak{C}_1(x)$ as seen (9) in (36), we obtain desired result given in (16). In Theorem 2.5, we have the following results for $\delta=1,\zeta=1$and $n=0$

Corollary 2.6 *If* $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta, n; \mathcal{R}\left(x, z\right)\right)$ *given in Example2.2, then*

$$|a_2| \leq \frac{3\left|x\right|\sqrt{3x}}{\sqrt{9x^2(1+2\zeta)3^n - \left(18x^2-1\right)\left(1+\zeta\right)^2 2^{2n}}}.$$

and

$$|a_3| \leq 3\left|x\right|\left[\frac{3}{2^{2n}\left(1+\zeta\right)^2}\left|x\right| + \frac{1}{3^n\left(1+2\zeta\right)}\right],$$

where $x \in \mathbb{C} \backslash \left\{ \mp \frac{2^n}{3} \frac{(1+\zeta)}{\sqrt{2^{2n+1}(1+\zeta)^2 - 3^n(1+2\zeta)}} \right\}$.

Corollary 2.7 *If* $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\mathcal{R}\left(x, z\right)\right)$ *defined inExample 2.2, then*

$$|a_2| \leq \frac{3\left|x\right|\sqrt{3x}}{\sqrt{27x^2 - 4\left(18x^2-1\right)}}.$$

and

$$|a_3| \leq 3\left|x\right|\left[\frac{3}{4}\left|x\right| + \frac{1}{3^{n+1}}\right],$$

where $x \in \mathbb{C} \backslash \left\{ \mp \frac{2\sqrt{5}}{15} \right\}$.

Corollary 2.8 *If* $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta, \delta; \mathcal{R}\left(x, z\right)\right)$ *defined in Example 2.3, then*

$$|a_2| \leq \frac{3\left|x\right|\sqrt{3x}}{\sqrt{9x^2(1+2\zeta) - \left(18x^2-1\right)\left(1+\zeta\right)^2}}.$$

and

$$|a_3| \leq 3\left|x\right|\left[\frac{3}{\left(1+\zeta\right)^2}\left|x\right| + \frac{1}{\left(1+2\zeta\right)}\right],$$

where $x \in \mathbb{C} \backslash \left\{ \mp \frac{1}{3} \frac{(1+\zeta)}{\sqrt{(1+\zeta)^2 + \zeta^2}} \right\}$.

Corollary 2.9 *If* $\mathfrak{K} \in \mathfrak{C}_{\Sigma}^{\mathcal{LB}}(1, \delta, n; \mathcal{R}(x, z)) = \mathfrak{C}_{\Sigma}^{\mathcal{LB}}(\delta, n; \mathcal{R}(x, z))$ *defined in Example 2.4, then*

$$|a_2| \leq \frac{3\left|x\right|\sqrt{3x}}{\sqrt{27x^2(1+2\delta)^n - 4\left(18x^2-1\right)\left(1+\delta\right)^{2n}}}.$$

and

$$|a_3| \leq 3\left|x\right|\left[\frac{3}{4(1+\delta)^{2n}}\left|x\right| + \frac{1}{3(1+2\delta)^n}\right],$$

where $x \in \mathbb{C} \backslash \left\{ \mp \frac{2}{3} \frac{(1+\delta)^n}{\sqrt{8(1+\delta)^{2n} - 3(1+2\delta)^n}} \right\}$.

3 Conclusion

In this study, a new subclass $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$ of bi- univalent functions are introduced by help of Al-Oboudi differential operator and principle of subordination, also via Lucas- Balancing polynomials. After than, we examined step by step the initial coefficients of functions belonging $\mathfrak{C}_{\Sigma}^{\mathcal{LB}}\left(\zeta,\delta,n;\mathcal{R}\left(x,z\right)\right)$.

4 Open Problem

We hope that this work encourage the researchers to obtain other characterization properties and relevant connections in other classes of univalent functions.

As an open problem, we can point out the following:

i) Fekete-Szegö inequality and Hankel determinant for this defined class can be investigated by

other researchers.

ii) Radii of starlikeness can be investigated in this class.

iii) A new class of $m-fold$ symmetric analytic functions introcuded by using properties of this class.

Conflict of interest Author declare that there is not any conflict of interests concerning the publication of this manuscript.

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