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A Comprehensive Examination of Probability Statements with Four Random Variables

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Abstract

This note presents three new problem statements involving random variables. These problems address the complexities and uncertainties inherent in real-world scenarios, making them valuable for modeling and analyzing diverse phenomena. A proposal for a solution is given to each of them. An open problem follows from the third problem statement.

Keywords: *Random variables, Independence, Sum of random variables.*
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1 Introduction

In this note, we explore three distinct problem statements involving four random variables. Random variables are essential components in probability and statistics, representing uncertain quantities that can take various values with certain probabilities. Understanding the relationships and dependencies between these random variables is crucial in various fields, such as finance, economics, and scientific research. See [1, 2, 3, 4, 5]. The three problem statements presented below offer intriguing challenges in analyzing and modeling the interactions between four random variables.

The plan of the note is as follows: Sections 2, 3 and 4 correspond to problem statements 1, 2, and 3, respectively. Section 5 is devoted to an open problem.

2 Problem Statement 1

The first problem statement is formulated as follows:

Statement of the problem 1. Give an example of four random variables $X \geq 0$, $Y \in \mathbb{R}$, $W \geq 0$, and $Z \leq 0$ such that
(i) X and Y are independent
(ii) $Cov(W, Z) < 0$
(iii) The distributions of $X + Y$ and $W + Z$ are identical.

A solution. Let X and Y be independent, with $X \geq 0$ and Y a random variable defined on $\mathbb{R}$ symmetric around 0. Define

$$W = X + |Y|, \quad Z = Y - |Y|.$$

Clearly $W \geq 0$ and $Z \leq 0$ since $Y \leq |Y|$. As for the other requirements,
(i) X and Y are independent by choice;
(ii) Thanks to the symmetric nature of Y, we have $Cov(Y, |Y|) = 0$. Therefore, a simple computation with expectations yields that

$$\begin{aligned} Cov(W, Z) &= Cov(X + |Y|, Y - |Y|) = Cov(|Y|, Y - |Y|) \\ &= Cov(Y, |Y|) - Var(|Y|) = -Var(|Y|) < 0; \end{aligned}$$

(iii) The distributions of $X + Y$ and $W + Z$ are identical because $X + Y = X + |Y| + Y - |Y| = W + Z$.

This ends the development. □

3 Problem Statement 2

The second problem statement is a variation of the first one, with nuances in the possible values of the involved random variables.

Statement of the problem 2. Give an example of four random variables $X, Y, W, Z \geq 0$ such that

(i) X and Y are independent
(ii) $Cov(W,Z) < 0$
(iii) The distributions of $X+Y$ and $W+Z$ are identical.

A solution. Let X and Y be independent, with $X \geq 0$ and Y uniform on $(0,1)$. Define

$$W = X + Y^\alpha, \quad Z = Y - Y^\alpha,$$

with $\alpha > 1$. Clearly $W \geq 0$ and $Z \geq 0$, the latter because $Y^\alpha \leq Y$ on $(0,1)$, so all the variables are nonnegative. As for the other requirements,
(i) X and Y are independent by choice;
(ii) A simple computation with expectations yields that

$$\begin{aligned} Cov(W,Z) &= Cov(Y,Y^\alpha) - Var(Y^\alpha) \\ &= E(Y^{\alpha+1}) - E(Y^\alpha)E(Y) - E(Y^{2\alpha}) + (E(Y^\alpha))^2 \\ &= -\frac{\alpha(\alpha-1)}{2(\alpha+1)^2(\alpha+2)(2\alpha+1)} < 0; \end{aligned}$$

(iii) The distributions of $X+Y$ and $W+Z$ are identical because $X+Y = X+Y^\alpha+Y-Y^\alpha = W+Z$.
The stated result is proved. □

Remarks. We notice that X does not play a role in the computation of $Cov(W,Z)$ and can be arbitrary as long as $X \geq 0$. Also, Y^α can be replaced by $f(Y)$, for any convex function in $(0,1)$ such that $Cov(Y, f(Y)) - Var(f(Y)) < 0$.

4 Problem Statement 3

The third and last problem statement adds a probability constraint to the second one, making it more complex to handle.

Statement of the problem 3. Give an example of four random variables $X, Y, W, Z \geq 0$ such that
(i) X and Y are independent
(ii) $Cov(W,Z) < 0$
(iii) The distributions of $X+Y$ and $W+Z$ are identical
(iv) $P(X+Y = W+Z) = 0$.

A solution. Clearly, the example in statement 2 does not satisfy condition (iv), so we look at the problem from a different angle. Let X and Y now be independent uniform (0,1) variables.

(i) X and Y are independent by definition;

(ii) Consider now the pair (W, Z), independent of the pair (X, Y), with uniform density on the triangle with vertices at $(0, 0)$, $(0, 2)$ and $(1, 0)$. It is clear that $W \geq 0$ and $Z \geq 0$. We have

$$E(WZ) = \frac{1}{6}, \quad E(W) = \frac{1}{3}, \quad E(Z) = \frac{2}{3},$$

so that

$$Cov(W, Z) = E(WZ) - E(W)E(Z) = -\frac{1}{18} < 0.$$

The condition (ii) is satisfied;

(iii) The density of $W + Z$ at x is given by the integral along straight line $z = x - w$ of the joint density function $f_{W,Z}$, that is:

$$f_{W+Z}(x) = \int_0^x f_{W,Z}(w, x - w)dw. \tag{1}$$

Let us consider the case $0 \leq x \leq 1$. Then the integral (1) becomes

$$\int_0^x dw = x.$$

On the other hand, if $1 \leq x \leq 2$, we only need to integrate from 0 to the w-coordinate of the intersection point of the lines $z = x - w$ and $z = -2w + 2$, i.e., from $w = 0$ to $w = 2 - x$, and the integral (1) becomes

$$\int_0^{2-x} dw = 2 - x.$$

Therefore we see that $W + Z$ has the same tent density as $X + Y$. the condition (iii) is satisfied;

(iv) It is well known that the density of $X + Y$ is a "tent" function

$$f_{X+Y}(z) = z, \quad 0 \leq z \leq 1,$$

and

$$f_{X+Y}(z) = 2 - z, \quad 1 \leq z \leq 2.$$

By the definition of (W, Z), it is clear that $X + Y \neq W + Z$ almost surely, implying the condition (iv).

The proposed solution is valuable. □

In conclusion, we have explored three new problem statements involving four random variables. These problem statements have provided us with valuable insights into different statistical scenarios and their implications. An open problem is also given with regard to the most technical problem statement.

5 Open Problem

Can we construct other examples that satisfy problem statement 3 beyond the use of the tent function ? The answer seems more complicated to solve than it seems at first glance.

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