Int. J. Open Problems Compt. Math., Vol. 16, No. 4, December 2023
Print ISSN: 1998-6262, Online ISSN: 2079-0376

A Fractional BVP Problem and Some Travelling Waves

Mahdi RAKAH, Zoubir DAHMANI and Yazid GOUARI

Laboratory of Pure and Applied Mathematics, University of Mostaganem,
Department of Mathematics, University of Alger 1, Algeria.
e-mail: mahdi.rakah@gmail.com
Department of Maths, Faculty of Sciences, University Blida 1,
Laboratory LMPA, University of Mostaganem, Algeria.
e-mail: zzdahmani@yahoo.fr
Ecole Normale Superieure, University of Mostaganem, Mostaganem, Algeria.
e-mail: gouariyazid@gmail.com

Received 1 November 2023; Accepted 10 December 2023

Abstract

We propose an alpha-beta-point non-linear boundary value problem that involves both Caputo derivatives and RL integrals, such that its limiting-case on the parameters alpha and beta product a fourth order problem that arises in bridge design. We begin first by presenting a uniqueness of solution result for the problem. Then, the Ulam-Hyers stability is discussed. An example is then discussed. Also, in a second part, we apply the tanh method for finding travelling waves for a nonlinear problem that involves Khalil derivatives in time and space. Some graphs are plotted for the obtained travelling waves.

Keywords: *Caputo derivative, fixed point, Riemann-Liouville integral, travelling waves.*

2010 Mathematics Subject Classification: 30C45. 39B72. 34A08. 35R11.

1 Introduction

Fractional differential equations theory is attracting more popularity and increasing importance, due to its numerous applications in various areas, such as optics, medicine, statistical physics, automatics, and control theory, see [5, 6, 10, 11, 14]. In this context, many authors have been interested in studying the question of the existence, uniqueness and stability of solutions for certain types of such equations. We refer the interested reader to [12, 13, 16]. Let us now on the other hand recall some papers that have motivated the present work. We begin by the references [26] where N. Urus et al. have studied the existence of solutions for a fourth order four-point non-linear boundary value problem (BVP) which arises in bridge design:

$$\begin{cases} -y^{(4)}(s) - \lambda y''(s) = f(s, y(s)), s \in (0,1), \\ y(0) = 0, \\ y''(0) = 0, \\ y(1) = \delta_1 y(\eta_1) + \delta_2 y(\eta_2), \\ y''(1) = \delta_1 y''(\eta_1) + \delta_2 y''(\eta_2), \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R}), \delta_1, \delta_2 > 0, 0 < \eta_1 \leq \eta_2 < 1, \lambda = \eta_1 + \eta_2$, η_1 and η_2 are real constants.

Also in [4], the authors have been concerned with a suitable fractional presentation for a simple Jerk circuit that allows us to study chaotic dynamics which was modeled by the problem:

$$\begin{cases} D^{\alpha}\left(D^2 + \lambda^2 D^{\alpha}\right) y(t) = f\left(t, y(t), D^{\alpha} y(t)\right), \quad t \in [0,T], \quad T > 0, \\ y(0) = 0, \\ D^{1-\alpha} D^{\alpha} y(0) = 0, \\ y(T) = \beta J^{\gamma} y(\eta), \quad 0 < \eta \leq T, \end{cases}$$

where D^{α} is Caputo fractional derivative of order $\alpha \in [0,1]$, J^{γ} is the Riemann-Liouville fractional integral order $\gamma \in [0, \infty[$, $\lambda \in \mathbb{R}_+$and $\beta \in \mathbb{R}$.

Very recently, A. Abdelnebi and Z. Dahmani [2] have studied the existence, uniqueness, and stability of solutions for the following Van der Pol-Duffing jerk equation:

$$\begin{cases} D^{\alpha}\left(D^{2-\beta} + \lambda D^{\alpha}\right) x(t) + k_1 f_1\left(t, x(t), D^{\alpha} x(t)\right) + k_2 f_2\left(t, x(t), J^p x(t)\right) = h(t). \\ x(1) = 0, \quad D^{1-(\alpha-\beta)} D^{\alpha-\beta} x(1) = A^* \in \mathbb{R}, \quad x(T) = 0, \\ 0 \leq \beta < \alpha \leq 1, \quad 0 \leq \alpha + \beta < 1, \quad 0 < p, \quad t \in I, \end{cases}$$

where $D^{\alpha}, D^{2-\beta}$, are the Caputo-Hadamard fractional derivatives, J^p is the Hadamard fractional integral $I = [1,T], k_1, k_2$ are real constants, and the functions f_1, f_2 and h are continuous.

Here, we begin first by studying the following problem:

$$\begin{cases} -D^{\beta}(D^{\alpha}+\lambda)y(t) = m(t, y(t), D^{\vartheta}y(t)) + n(t, y(t), I^{p}y(t)) + r(t, y(t)) + l(t), t \in J, \\ y(0) = 0, \\ D^{\alpha}y(0) = 0, \\ y(1) = ay(\xi_1) + by(\xi_2), \\ D^{\alpha}y(1) = a^{'}D^{\alpha}y(\xi_1) + b^{'}D^{\alpha}y(\xi_2), \end{cases} \tag{1}$$

under the fact that $J := [0,1]$, $a, a^{'}, b, b^{'} \in \mathbb{R}, 0 < \xi_1 \leq \xi_2 < 1, \lambda = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are the real constants, $0 < \vartheta \leq 1$, $1 < \alpha, \beta \leq 2$, and m, n, r, and l are some functions that will be specified later, D^{α}, D^{β} and D^{ϑ} are the derivatives in the sense of Caputo.

In the second part of our paper, we will be concerned with some applications of the tanh method to find new traveling wave solutions for some special equations. We shall note that the phenomena of traveling waves are observed in a several applied fields, and several techniques have been successfully developed in this sense, such as the exp-function method [3, 15, 23], the (G'/G) method [28], and the tanh method. This method is one of the most effective algebraic methods for finding solutions to nonlinear differential equations. It was presented by Malfliet [21] and then modified and extended by Wazwaz [27].

For the above-motivating method and to cite some of the papers that have motivated the present part, we begin by the reference [24], where, M. Rakah et al. have been concerned with finding traveling wave solutions for the following fractional evolution problem [19]:

$$T_t^{2\alpha}u + T_x^{\beta}(G(u)T_x^{3\beta}u) + T_x^{\beta}(H(u)T_x^{\beta}u) = F(u),$$

where, $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivatives, with $0 < \alpha, \beta \leq 1$ and f, G, H are some given functions.

Very recently in [9], Z. Dahmani et al. have presented an $(n{+}1)$-dimensional extended tanh function method to investigate nonlinear problems.

So in particular, in our second part, we will use the tanh method to find new traveling wave solutions of the following problem with its unknown function u that depends on time and space:

$$T_{2\alpha}^{t}u + T_{4\beta}^{x}u + H(u) = 0, \tag{2}$$

taking into account that $T_{\alpha}^{t}, T_{\beta}^{x}$ are the conformable fractional derivative; $0 < \alpha, \beta \leq 1$, and H is a continuous function.

It is important to say that the above two proposed problems (1) and (2) may have a connection between them since both of them can induce a beam type equation of fourth order as a particular limiting case.

2 Preliminaries

In this section, we have to present to the reader definitions and properties on fractional calculus, see [20, 22].

Definition 2.1 *Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2 *For any $f \in C^n(J,\mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo derivative is defined by:*

$$\begin{aligned} D^\alpha f(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

Lemma 2.3 *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then, the set of solutions of $D^\alpha y(t) = 0; t \in J$ is given by the elements:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i,$$

such that $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Lemma 2.4 *If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then, one has*

$$I^\alpha D^\alpha y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i,$$

under the fact that $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Let us prove the following lemma.

Lemma 2.5 *We consider $K \in C(I)$. Then the BVP*

$$\begin{cases} -D^\beta (D^\alpha + \lambda) y(t) = K(t), t \in [0,1], \\ y(0) = 0, \\ D^\alpha y(0) = 0, \\ y(1) = a y(\xi_1) + b y(\xi_2), \\ D^\alpha y(1) = a' D^\alpha y(\xi_1) + b' D^\alpha y(\xi_2), \\ 0 < \alpha, \beta \leq 2, \end{cases} \tag{3}$$

is equivalent to:

$$\begin{aligned}
y(t) &= -I^{\alpha+\beta}K(t)+\lambda I^{\alpha}y(t)-\Delta_1^{-1}\Bigg[I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K(\xi_2)\\
&+ b^{'}\lambda y(\xi_2)\Bigg]\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\Delta_3^{-1}\Bigg[I^{\alpha+\beta}K(1)-\lambda I^{\alpha}y(1)-aI^{\alpha+\beta}K(\xi_1)+a\lambda I^{\alpha}y(\xi_1)\\
&- bI^{\alpha+\beta}K(\xi_2)+b\lambda I^{\alpha}y(\xi_2)-\Delta_2\Delta_1^{-1}\Bigg(I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)\\
&- b^{'}I^{\beta}K(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg)\Bigg]t,
\end{aligned}$$

such that

$$\Delta_1 = a^{'}\xi_1+b^{'}\xi_2-1,$$

$$\Delta_2 = \frac{a\xi_1^{\alpha+1}+b\xi_2^{\alpha+1}-1}{\Gamma(\alpha+2)},$$

$$\Delta_3 = a\xi_1+b\xi_2-1,$$

and,

$$\Delta_1\Delta_3 \neq 0.$$

Proof.
Thanks to Lemma 2.4, it yields

$$D^{\alpha}y(t)+\lambda y(t) = I^{\beta}K(t)+c_0t+c_1.$$

Therefore,

$$y(t) = -I^{\alpha+\beta}K(t)+\lambda I^{\alpha}y(t)-c_0\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-c_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}-c_2t-c_3. \tag{4}$$

The conditions

$$y(0)=0, \qquad D^{\alpha}y(0)=0,$$

allow us to write

$$c_1=c_3=0,$$

and the conditions

$$y(1)=ay(\xi_1)+by(\xi_2),$$

and

$$D^{\alpha}y(1)=a^{'}D^{\alpha}y(\xi_1)+b^{'}D^{\alpha}y(\xi_2),$$

allow us to get

$$\begin{aligned} c_0 &= \Delta_1^{-1}\left[I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K(\xi_2)+b^{'}\lambda y(\xi_2)\right], \\ c_2 &= \Delta_3^{-1}\Bigg[I^{\alpha+\beta}K(1)-\lambda I^{\alpha}y(1)-aI^{\alpha+\beta}K(\xi_1)+a\lambda I^{\alpha}y(\xi_1)-bI^{\alpha+\beta}K(\xi_2)+b\lambda I^{\alpha}y(\xi_2) \\ &- \Delta_2\Delta_1^{-1}\left(I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K(\xi_2)+b^{'}\lambda y(\xi_2)\right)\Bigg]. \end{aligned}$$

The proof is thus achieved.

3 Main results

Let consider the space

$$N:=\{y\in C(J,\mathbb{R}),D^{\vartheta}y\in C(J,\mathbb{R})\},$$

and the norm:

$$\|y\|_N=\|y\|_{\infty}+\|D^{\vartheta}y\|_{\infty},$$

where,

$$\|y\|_{\infty}=\sup_{t\in J}|y(t)|\ ,\|D^{\vartheta}y\|_{\infty}=\sup_{t\in J}|D^{\vartheta}y(t)|.$$

Then, we take the nonlinear operator $Z:N\to N$, where:

$$\begin{aligned} Zy(t) &= -I^{\alpha+\beta}K_y^*(t)+\lambda I^{\alpha}y(t)-\Delta_1^{-1}\Bigg[I^{\beta}K_y^*(1)-\lambda y(1)-a^{'}I^{\beta}K_y^*(\xi_1)+a^{'}\lambda y(\xi_1) \\ &- b^{'}I^{\beta}K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg]\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\Delta_3^{-1}\Bigg[I^{\alpha+\beta}K_y^*(1)-\lambda I^{\alpha}y(1)-aI^{\alpha+\beta}K_y^*(\xi_1) \\ &+ a\lambda I^{\alpha}y(\xi_1)-bI^{\alpha+\beta}K_y^*(\xi_2)+b\lambda I^{\alpha}y(\xi_2)-\Delta_2\Delta_1^{-1}\Bigg(I^{\beta}K_y^*(1)-\lambda y(1)-a^{'}I^{\beta}K_y^*(\xi_1) \\ &+ a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg)\Bigg]t, \end{aligned}$$

where

$$K_y^*(t)=m(t,y(t),D^{\vartheta}y(t))+n(t,y(t),I^{p}y(t))+r(t,y(t))+l(t).$$

The above notions are introduced in order to transform our problem into a one of fixed point.

The following hypotheses are only sufficient; one can use other conditions of Caratheodory functions to obtain the same results.
$(\Phi 1)$: The introduced functions are continuous.

$(\Phi 2)$: There exist nonnegative constants $\chi_{m1}, \chi_{m2}, \chi_{n1}, \chi_{n2}$, such that for any t $\in J$, $y_i, y_i^* \in \mathbb{R}$,

$$|m(t, y_1, y_2) - m(t, {y_1}^*, {y_2}^*)| \leq \sum_{i=1}^{2} \chi_{mi}|y_i - {y_i}^*|,$$

$$|n(t, y_1, y_2) - n(t, {y_1}^*, {y_2}^*)| \leq \sum_{i=1}^{2} \chi_{ni}|y_i - {y_i}^*|.$$

and for any t $\in J$, $y, y^{'} \in \mathbb{R}$,

$$|r(t, y) - r(t, y^{'})| \leq Q|y - y^{'}|.$$

We put in passage:

$$X := Max(\chi_{m1}, \chi_{m2}), X^* := Max(\chi_{n1}, \chi_{n2}).$$

Also, the quantities

$$\theta_1 = \frac{1}{\Gamma(\alpha+\beta+1)} + |\Delta_3^{-1}| \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)},$$

$$\varphi_1 = \frac{1}{\Gamma(\alpha+1)} + |\Delta_3^{-1}| \times \Big(1+|a|+|b|\Big) + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)\Big(1+|a'|+|b'|\Big),$$

$$\theta_2 = \frac{1}{\Gamma(\alpha+\beta-\vartheta+1)} + \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)},$$

and

$$\varphi_2 = \frac{1}{\Gamma(\alpha+1)} + \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \Big(1+|a|+|b|\Big) + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right) \times \Big(1+|a'|+|b'|\Big),$$

are to be considered in this paper

3.1 First result

We propose to establish the following result:

Theorem 3.1 *Assume that* $(\Phi1), (\Phi2)$ *are satisfied. Then, (1) has a unique solution if* $\Sigma(\theta_1+\theta_2) < 1-\lambda(\varphi_1+\varphi_2)$*;* $\Sigma = Q + 2X + X^* + \dfrac{X^*}{\Gamma(p+1)}$.

Proof.
Let us take $(y, y^{'}) \in N^2$, we can write

$$\begin{aligned}\|Zy - Zy^{'}\|_\infty &\le \Sigma\left[\frac{1}{\Gamma(\alpha+\beta+1)} + |\Delta_3^{-1}| \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)\right.\\ &\times \left.\frac{1+|a'|+|b'|}{\Gamma(\beta+1)}\right]\|y-y^{'}\|_N + \lambda\left[\frac{1}{\Gamma(\alpha+1)} + |\Delta_3^{-1}| \times \Big(1+|a|+|b|\Big)\right.\\ &+ \left.\left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)\Big(1+|a'|+|b'|\Big)\right]\|y-y^{'}\|_N.\end{aligned}$$

The reader can see also that

$$\begin{aligned}
D^{\vartheta}Zy(t) &= -I^{\alpha+\beta-\vartheta}K_y^*(t)+\lambda I^{\alpha-\vartheta}y(t)-\Delta_1^{-1}\Bigg[I^{\beta}K_y^*(1)-\lambda y(1)-a^{'}I^{\beta}K_y^*(\xi_1)+a^{'}\lambda y(\xi_1)\\
&- b^{'}I^{\beta}K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg]\frac{t^{\alpha-\vartheta}}{\Gamma(\alpha-\vartheta+1)}-\Delta_3^{-1}\Bigg[I^{\alpha+\beta}K_y^*(1)-\lambda I^{\alpha}y(1)-aI^{\alpha+\beta}K_y^*(\xi_1)\\
&+ a\lambda I^{\alpha}y(\xi_1)-bI^{\alpha+\beta}K_y^*(\xi_2)+b\lambda I^{\alpha}y(\xi_2)-\Delta_2\Delta_1^{-1}\Bigg(I^{\beta}K_y^*(1)-\lambda y(1)-a^{'}I^{\beta}K_y^*(\xi_1)\\
&+ a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg)\Bigg]\frac{t^{1-\vartheta}}{\Gamma(2-\vartheta)},
\end{aligned}$$

and

$$\begin{aligned}
\|D^{\vartheta}Zy-D^{\vartheta}Zy^{'}\|_{\infty} &\leq \Sigma\Bigg[\frac{1}{\Gamma(\alpha+\beta-\vartheta+1)}+\frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)}\times\frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)}+\Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)}\\
&+ |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg)\times\frac{1+|a'|+|b'|}{\Gamma(\beta+1)}\Bigg]\|y-y^{'}\|_N+\lambda\Bigg[\frac{1}{\Gamma(\alpha+1)}\\
&+ \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)}\times\Big(1+|a|+|b|\Big)+\Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)}+|\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg)\\
&\times \Big(1+|a'|+|b'|\Big)\Bigg]\|y-y^{'}\|_N.
\end{aligned}$$

Therefore,

$$\|Zy-Zy^{'}\|_N \leq \Bigg[\Sigma(\theta_1+\theta_2)+\lambda(\varphi_1+\varphi_2)\Bigg]\|y-y^{'}\|_N.$$

Example 3.2 *We consider the following problem:*

$$\begin{cases} -D^{\frac{4}{3}}(D^{\frac{3}{2}}+\frac{1}{20})y(t)=\frac{1}{100}\Big(e^{t-4}\sin(y(t))+\frac{1}{e^{t+2}}D^{\frac{1}{2}}y(t)+\frac{\sin(t+1)}{2}\Big)\\ +10^{-1}\Big(\frac{1}{25}y(t)+\frac{\cos(3+t^2)}{\pi(10+t)}+\frac{1}{30}I^{\frac{1}{2}}y(t)\Big)\\ +\frac{1}{40}\Big(\frac{1}{\pi^2}y(t)+ln(t+1)\Big)+\frac{3}{2}t,\\ y(0)=0,\\ D^{\frac{3}{2}}y(0)=0,\\ y(1)=\frac{1}{10}y(\frac{7}{200})+\frac{1}{5}y(\frac{3}{200}),\\ D^{\frac{3}{2}}y(1)=\frac{1}{5}D^{\alpha}y(\frac{7}{200})+\frac{3}{10}D^{\frac{3}{2}}y(\frac{3}{200}), \end{cases}$$

with

$$m(t,y_1,y_2)=\frac{1}{100}\Big(e^{t-4}\sin(y(t))+\frac{1}{e^{t+2}}D^{\frac{1}{2}}y(t)+\frac{\sin(t+1)}{2}\Big),$$
$$n(t,y_1,y_2)=10^{-1}\Big(\frac{1}{25}y(t)+\frac{\cos(3+t^2)}{\pi(10+t)}+\frac{1}{30}I^{\frac{1}{2}}y(t)\Big),$$
$$r(t,y)=\frac{1}{40}\Big(\frac{1}{\pi^2}y(t)+ln(t+1)\Big),$$
$$l(t)=\frac{3}{2}t,$$
$$\Sigma=0.0138,\quad \theta_1=3.4010,\quad \theta_2=5.0352,\quad \lambda=\frac{1}{20},\quad \varphi_1=5.3492,\quad \varphi_2=5.8544.$$

So, we see that

$$\Sigma(\theta_1+\theta_2)<1-\lambda(\varphi_1+\varphi_2).$$

The conditions of Theorem 6 *hold. Therefore, the problem has a unique solution over* J.

3.2 Second result

Definition 3.3 *The equation* (1) *has the Ulam Hyers stability if there is a* $\Lambda>0$*; for each* $\varpi>0, t\in[0,1]$ *and for each* $y\in T$ *solution of*

$$\big|D^{\beta}(D^{\alpha}+\lambda)y(t)+m(t,y(t),D^{\vartheta}y(t))+n(t,y(t),I^{p}y(t))+r(t,y(t))+l(t)|\leq\varpi, \tag{5}$$

there is certainly $y^*\in T$ *a solution of* (1)*;*

$$\|y-y^*\|_N\leq\Lambda\varpi.$$

Definition 3.4 *The equation* (1) *has the Ulam Hyers stability in the generalized sense if there is* $\rho\in C(\mathbb{R}^+,\mathbb{R}^+)$*;* $\rho(0)=0$*; for each* $\varpi>0$*, and for any* $y\in T$ *solution of* (5)*, there is a solution* $y^*\in N$ *of (1);*

$$\|y-y^*\|_N<\rho(\varpi).$$

We prove the following theorem

Theorem 3.5 *Under the conditions of Theorem* 3.1, *problem* (1) *is Ulam Hyers stable.*

Proof: Let $y \in N$ be a solution of (5), and let $y^* \in N$ be the unique solution of (1).
We have:

$$\Bigg| y(t) + I^{\alpha+\beta} K_y^*(t) - \lambda I^\alpha y(t) + \Delta_1^{-1}\Bigg[I^\beta K_y^*(1) - \lambda y(1) - a^{'} I^\beta K_y^*(\xi_1) + a^{'}\lambda y(\xi_1) - b^{'} I^\beta K_y^*(\xi_2)$$

$$+ b^{'}\lambda y(\xi_2)\Bigg] \times \frac{t^\alpha}{\Gamma(\alpha+1)} + \Delta_3^{-1}\Bigg[I^{\alpha+\beta} K_y^*(1) - \lambda I^\alpha y(1) - a I^{\alpha+\beta} K_y^*(\xi_1) + a\lambda I^\alpha y(\xi_1) - b I^{\alpha+\beta} K_y^*(\xi_2)$$

$$+ b\lambda I^\alpha y(\xi_2) - \Delta_2\Delta_1^{-1}\Bigg(I^\beta K_y^*(1) - \lambda y(1) - a^{'} I^\beta K_y^*(\xi_1) + a^{'}\lambda y(\xi_1) - b^{'} I^\beta K_y^*(\xi_2) + b^{'}\lambda y(\xi_2)\Bigg)\Bigg] t \Bigg|$$

$$\leq \frac{\varpi}{\Gamma(\alpha+\beta+1)}.$$

By (5) and (3.2), we can write

$$\|y - y^*\|_\infty \leq \frac{\varpi}{\Gamma(\alpha+\beta+1)} + \Sigma\Bigg[\frac{1}{\Gamma(\alpha+\beta+1)} + |\Delta_3^{-1}| \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)}$$

$$+ |\Delta_1^{-1}\Delta_2\Delta_3^{-1}| \Bigg) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)} \Bigg] \|y - y^*\|_N + \lambda\Bigg[\frac{1}{\Gamma(\alpha+1)} + |\Delta_3^{-1}| \times \Big(1+|a|+|b|\Big)$$

$$+ \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}| \Bigg) \Big(1+|a'|+|b'|\Big) \Bigg] \|y - y^*\|_N. \tag{6}$$

Hence,

$$\|y - y^*\|_\infty \leq \frac{\varpi}{\Gamma(\alpha+\beta+1)} + (\Sigma\theta_1 + \lambda\varphi_1)\|y - y^*\|_N.$$

Also, the reader can observe that

$$\|D^\vartheta(y - y^*)\|_\infty \leq \frac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)} + (\Sigma\theta_2 + \lambda\varphi_2)\|y - y^*\|_N.$$

Thus,

$$\|y-y^*\|_N \leq \left(\frac{\varpi}{\Gamma(\alpha+\beta+1)}+\frac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)}\right)+\left[\Sigma(\theta_1+\theta_2)+\lambda(\varphi_1+\varphi_2)\right]\|y-y^*\|_N.$$

$$\|y-y^*\|_N \leq \frac{\dfrac{\varpi}{\Gamma(\alpha+\beta+1)}+\dfrac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)}}{1-\left[\Sigma(\theta_1+\theta_2)+\lambda(\varphi_1+\varphi_2)\right]}.$$

So,

$$\|y-y^*\|_N \leq \Lambda\varpi.$$

Consequently, (1) has the Ulam Hyers stability.

Remark 3.6 *The case* $\rho(\varpi)=\Lambda\varpi$ *allows us to guarantee the generalised Ulam Hyers stability for* (1)*.*

4 Second part

Let us consider the following problem:

$$T^t_{2\alpha}u+T^x_{4\beta}u+H(u)=0,$$

where, T^t_α, T^x_β are the conformable fractional derivative, with $0<\alpha,\beta\leq 1$. Taking $\alpha=\beta=1$, the above problem is transformed into the so called Beam equation:

$$u_{tt}+u_{xxxx}+H(u)=0. \tag{7}$$

4.1 Conformable Fractional Derivatives

In this subsection, we recall the following definitions [19].

Definition 4.1 *Given a function* $f:(0,\infty)\to\mathbb{R}$.

Then, the conformable fractional derivative of order α *is defined by*

$$T_\alpha(f)(t)=\frac{\partial^\alpha f(t,x)}{\partial t^\alpha}=\lim_{\varepsilon\to 0}\left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}\right),\quad t>0,\quad 0<\alpha\leq 1.$$

Several properties can be given:

$$\begin{aligned} T_\alpha(af+bg) &= aT_\alpha(f)+bT_\alpha(g),\ \text{for all } a,b\in\mathbb{R},\\ T_\alpha(C) &= 0,\\ T_\alpha\left(t^b\right) &= bt^{b-\alpha},\ \text{for all } b\in\mathbb{R},\\ T_\alpha(fg) &= gT_\alpha(f)+fT_\alpha(g), \end{aligned}$$

if f is differentiable, $T_\alpha(f)(t) = t^{1-a}\frac{df}{dt}(t)$.

Definition 4.2 *The conformable fractional integral of a function $f:(0,\infty)\to\mathbb{R}$ of order α is defined as*

$$J_\alpha(t) = \int_0^t \tau^{\alpha-1} f(\tau)\,d\tau, \qquad 0<\alpha\le 1.$$

The following properties are important for our work.

$$J_\alpha T_\alpha f(t) = f(t) - f(0),$$

4.2 Tanh method

We recall the main steps of Tanh method for the case of Khalil derivatives [9].

Let us consider the nonlinear differential equation in the form

$$R\left(u, T_\alpha^t, T_\beta^x, T_{2\alpha}^t, T_\alpha^t(T_\beta^x), T_{2\beta}^x, ...\right) = 0, \tag{8}$$

where $T_\alpha^t u$ is the conformable fractional derivative of u of order $\alpha, 0<\alpha\le 1$.

Step1 We suppose that the wave variable is given by

$$\nu = \tfrac{k}{\alpha}t^\alpha + \tfrac{\omega}{\beta}x^\beta, u(x,t) = U(\nu), \tag{9}$$

where k and ω are constants,

It will allow us to transform (8) into:

$$R^*\left(U, U', U'', U''', ...\right) = 0, \tag{10}$$

Step2 Now, we take

$$\eta = tanh(\nu), \tag{11}$$

The variable (11) gives us:

$$\begin{array}{rcl} \frac{d}{d\nu} &=& \left(1-\eta^2\right)\frac{d}{d\eta}, \\ \frac{d^2}{d\nu^2} &=& -2\eta\left(1-\eta^2\right)\frac{d}{d\eta}+\left(1-\eta^2\right)^2\frac{d^2}{d\eta^2}, \\ \frac{d^3}{d\nu^3} &=& 2\left(1-\eta^2\right)\left(3\eta^2-1\right)\frac{d}{d\eta}-6Y\left(1-\eta^2\right)^2\frac{d^2}{d\eta^2}+\left(1-\eta^2\right)^3\frac{d^3}{d\eta^3}, \\ \frac{d^4}{d\nu^4} &=& -8\eta\left(1-\eta^2\right)\left(3\eta^2-2\right)\frac{d}{d\eta}+4\left(1-\eta^2\right)^2\left(9\eta^2-2\right)\frac{d^2}{d\eta^2} \\ && -12\eta\left(1-\eta^2\right)^3\frac{d^3}{d\eta^3}+\left(1-\eta^2\right)^4\frac{d^4}{d\eta^4}. \end{array} \tag{12}$$

Step3 If we assume that the solution can be expressed by

$$u(x,t)=U(\nu)=S(\eta)=\textstyle\sum_{i=0}^{m} b_i\eta^i, \tag{13}$$

where m is a positive integer determined by the balancing procedure in the resulting nonlinear ODE in S, we will find algebraic equations from which the constants $k,\omega,b_i(i=0,\cdots,m)$ are obtained.

4.3 An Example

Now we consider [17, 7]:

$$T^t_{2\alpha}u+T^x_{4\beta}u+(n_1+n_2u)u=0. \tag{14}$$

Using (9) to change (14) into the following problem

$$k^2U_{\nu\nu}+\omega^4U_{\nu\nu\nu\nu}+(n_1+n_2U)U=0.$$

Then, we have

$$k^2U_{\nu\nu}+\omega^4U_{\nu\nu\nu\nu}+n_1U+n_2U^2=0. \tag{15}$$

Substituting (12) and (13) into (15), we can get

$$k^2\Big[-2\eta(1-\eta^2)\frac{dS}{d\eta}+(1-\eta^2)^2\frac{d^2S}{d\eta^2}\Big]+\omega^4\Big[-8\psi(1-\eta^2)(3\eta^2-2)\frac{dS}{d\eta}$$
$$+4(1-\eta^2)^2(9\eta^2-2)\frac{d^2S}{d\eta^2}-12\psi(1-\eta^2)^3\frac{d^3S}{d\eta^3}+(1-\eta^2)^4\frac{d^4S}{d\eta^4}\Big]+n_1S+n_2S^2=0. \tag{16}$$

To determine the parameter m we use the balance of $\eta^8\frac{d^4S}{d\eta^4}$ with S^2. This implies:

$$8+m-4=2m$$

so $m=4$. The solution is in the form

$$S(\eta)=b_0+b_1\eta+b_2\eta^2+b_3\eta^3+b_4\eta^4. \tag{17}$$

Substituting (17) into (16), we can get

$$k^2\Big[-2\eta(1-\eta^2)(b_1+2b_2\eta+3b_3\eta^2+4b_4\eta^3)+(1-\eta^2)^2(2b_2+6b_3\eta+12b_4\eta^2)\Big]$$
$$+\omega^4\Big[-8\eta(1-\eta^2)(3\eta^2-2)(b_1+2b_2\eta+3b_3\eta^2+4b_4\eta^3)+4(1-\eta^2)^2(9\eta^2-2)$$
$$\times(2b_2+6b_3\eta+12b_4\eta^2)-12\eta(1-\eta^2)^3(6b_3+24b_4\eta)+24b_4(1-\eta^2)^4\Big]$$
$$+n_1(b_0+b_1\eta+b_2\eta^2+b_3\eta^3+b_4\eta^4)+n_2(b_0+b_1\eta+b_2\eta^2+b_3\eta^3+b_4\eta^4)^2=0. \tag{18}$$

Then, we have the system:

$$\begin{cases} \eta^0: -16\omega^4b_2+24\omega^4b_4++2k^2b_2+b_0^2n_2+b_0n_1=0, \\ \eta^1: 16\omega^4b_1-120\omega^4b_3-2k^2b_1+6b_3k^2+2b_0b_1n_2+b_1n_1=0, \\ \eta^2: 136\omega^4b_2-480\omega^4b_4-8k^2b_2+12k^2b_4+2b_0b_2n_2+b_1^2n_2+b_2n_1=0, \\ \eta^3: -40\omega^4b_1+576\omega^4b_3+2k^2b_1-18k^2b_3+2b_0b_3n_2+2b_1b_2n_2+b_3n_1=0, \\ \eta^4: -240\omega^4b_2+1696\omega^4b_4+6k^2b_2-32k^2b_4+2b_0b_4n_2+2b_1b_3n_2+b_2^2n_2+b_4n_1=0, \\ \eta^5: 24\omega^4b_1-816\omega^4b_3+12k^2b_3+2b_1b_4n_2+2b_2b_3n_2=0, \\ \eta^6: 120\omega^4b_2-2080\omega^4b_4+20k^2b_4+2b_2b_4n_2+b_3^2n_2=0, \\ \eta^7: 360\omega^4b_3+2b_3b_4n_2=0, \\ \eta^8: 840\omega^4b_4+b_4^2n_2=0. \end{cases} \tag{19}$$

We obtain the solutions of (14):

Case 1:

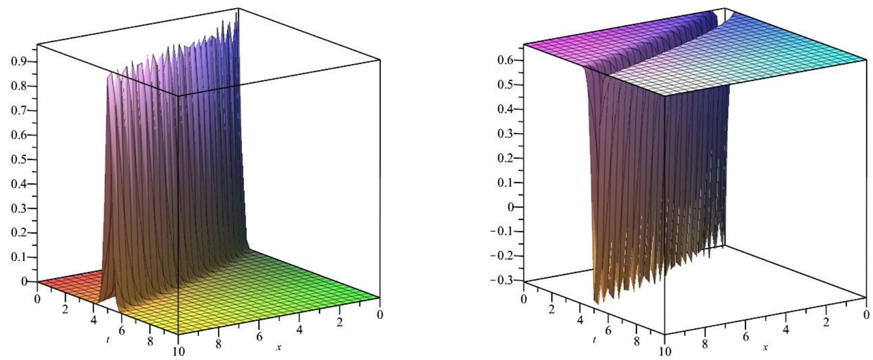
$$b_0=-\frac{35n_1}{24n_2}, b_1=0, b_2=\frac{35n_1}{12n_2}, b_3=0, b_4=-\frac{35n_1}{24n_2},$$

$$u(x,t)=-\frac{35n_1}{24n_2}+\frac{35n_1}{12n_2}\tanh^2(\xi)-\frac{35n_1}{24n_2}\tanh^4(\xi). \tag{20}$$

Case 2:

$$b_0=\frac{11n_1}{24n_2}, b_1=0, b_2=-\frac{35n_1}{12n_2}, b_3=0, b_4=\frac{35n_1}{24n_2},$$

$$u(x,t)=\frac{11n_1}{24n_2}-\frac{35n_1}{12n_2}\tanh^2(\xi)+\frac{35n_1}{24n_2}\tanh^4(\xi). \tag{21}$$



(a) Plots of solution (20) (b) Plots of solution (21)

Figure 1: *Plots of solution with* $0 \leq x \leq 10$, $0 \leq t \leq 30$ and $n_1 = -2, n_2 = 3, \alpha = \frac{9}{10}, \beta = \frac{7}{10}$.

5 Conclusion

We have introduced a new alpha-beta-order four-point non-linear boundary value problem where its limiting-case arises in design. Our problem involves Riemann-Liouville integrals and Caputo derivatives. A uniqueness of solution result has been first proved. The stabilities UH and GUH of solutions have also been presented. An illustrative example has been presented to show the applicability of the main results. Also, some results on travelling waves for an alpha, beta fractional partial differential equation, using Khalil approach and the tanh method, have been obtained and some of their graphs have been plotted.

6 Open Problem

Open question 1: Since closed boundary conditions are important in applications, so is it possible to consider a four sequential differential problem with such conditions? What about the existence of a unique solution in this case? Is there a possible comparison between the solutions of these two different cases?
Open question 2: Is it possible to compare the existence of a unique solution for the above considered problem with the standard case where the parameters are integer?

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math, 279 (2015), 57-66.

[2] A. Abdelnebi and Z. Dahmani, *New Van der Pol-Duffing Jerk Fractional Differential Oscillator of Sequential Type*, Mathematics, 10 (2022), 35-46.

[3] A.Anber, Z. Dahmani, *The SGEM Method For Solving Some Time and Space-Conformable Fractional Evolution Problems,* Int. J. Open Problems Compt. Math., 16(2023), 33-44.

[4] M. Belhamiti, Z. Dahmani, A. Praveen, *Chaotic Jerk Circuit: Existence and Stability of Solutions for a Fractional Model*, Progress in Fractional Differentiation and Applications 9(3) (2023).

[5] Y. Bahous , Z. Dahmani, *A Lane Emden Type Problem Involving Caputo Derivative and Riemann-Liouville Integral*, Indian Journal of Industrial and Applied Mathematics 10(1) (2019) 60-75.

[6] A. Carpinteri , F. Mainardi, Fractional Calculus in Continuum Mechanics, Springer, New York, NY. (1997).

[7] J. Chang, X. Zhao, L. H. Liu, X. Liang, The Traveling Wave Solutions of Nonlinear Beam Equations by Tanh-Function Method, Advanced Materials Research, 889-890, (2014), 628-632.

[8] Z. Dahmani, A. Anber, Y. Gouari, M. Kaid and I, Jebril, *Extension of a Method for Solving Nonlinear Evolution Equations Via Conformable Fractional Approach*, International Conference on Information Technology, (2021), 38-42.

[9] Z. Dahmani, A. Anber, and I. Jebril, *Solving conformable evolution equations by an extended numerical method*, Jordan Journal of Mathematics and Statistics (JJMS), 15(2) (2022), 363-380.

[10] Z. Dahmani, Y. Bahous and Z. Bekkouche, A two parameter singular fractional diferential equations of Lane Emden type, , Turkish J. Ineq. 3 (1) (2019), 35-53.

[11] Z. Dahmani, M.A. Abdellaoui, M. Houas, Coupled Systems of Fractional Integro-Diferential Equations Involving Several Functions, Theory and Applications of Mathematics and Computer Science, 5(1) (2015), 53-61.

[12] Y. Gouari, Z. Dahmani, I. Jebril, Application of fractional calculus on a new differential problem of duffing type, Adv. Math. Sci. J. 12 (2020) 10989-11002.

[13] Y. Gouari, M. Rakah, Z. Dahmani, *A sequential differetial problem with caputo and riemann liouville derivatives involving convergent series*, Adv. Theory Nonlinear Anal. Appl., 7(2) (2023), 319-335.

[14] Y. Gouari, Z. Dahmani, A. Ndiaye, A generalized sequential problem of Lane-Emden type via fractional calculus, Moroccan J. of Pure and Appl. Anal. 6 (2020) 168-183.

[15] J.H. He, Exp-function method for fractional differential equations, Int. J. Nonlin- ear Sci. Numer. Simul, 14(6) (2013), 363-366.

[16] M. Houas, M.E. Samei, Existence and Mittag-Leffler-Ulam-Stability Results for Duffing Type Problem Involving Sequential Fractional Derivatives, International Journal of Applied and Computational Mathematics, 8(4) (2022).

[17] M. A. A. Hussain, Two Modes Bifurcation Solutions of Elastic Beams Equation with Nonlinear Approximation, Communications in Mathematics and Applications journal India, 1 (2010), 123-131.

[18] G. Jumarie, *Modified riemann-liouville derivative and fractional taylor series of nondifferentiable functions further results*, Computers Mathematics with Applications, 51(9-10) (2006), 1367-1376.

[19] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264 (2014), 65-70.

[20] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B.V., Amsterdam, The Netherlands, (2006).

[21] W. Malfliet, *The tanh method: I. Exact solutions of nonlinear evolution and wave equations*, Physica Scripta, 54 (1996), 563-568.

[22] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York (1993).

[23] M. Rakah, A. Anber, Z. Dahmani and I. Jebril, *An Analytic and Numerical study for two classes of differential equation of fractional order involving Caputo and Khalil derivative,* An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 69(1) (2023), 29-48.

[24] M. Rakah, Z. Dahmani, A. Senouci, *New Uniqueness Results for Fractional Differential Equations with a Caputo and Khalil Derivatives.* Appl. Math. Inf. Sci., 16(6) (2022), 943-952.

[25] M. Rakah, Y. Gouari, R.W. Ibrahim, Z. Dahmani, H. Kahtan, *Unique solutions, stability and travelling waves for some generalized fractional differential problems.* Applied Mathematics in Science and Engineering, 23(1), (2023).

[26] N. Urus, A.K. Verma, *Existance and uniqueness results for fourth-order four-point BVP arising in bridge design in the presence of reverse ordered upper and lower solutions*, Electronic Journal of Differential Equations 2023(51) (2023).

[27] A.M. Wazwaz, *The tanh method for compact and non compact solutions for variants of the KdVBurger equations*, Phys. D: Nonlinear Phenomena, 213 (2006), 147-151.

[28] B. Zheng, *(G'/G)-Expansion method for solving fractional partial differential equations in the theory of mathematical physics*, Communications in Theoretical Physics, 58 (2012), 623-630.