

## Harmonic maps on $C_{12}$ -Manifolds

Beldjilali Gherici

Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M),  
University of Mascara, Algeria.  
e-mail: [gherici.beldjilali@univ-mascara.dz](mailto:gherici.beldjilali@univ-mascara.dz)

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### Abstract

*Motivated by the recent studies of the harmonic maps on normal almost contact metric manifolds, in particular Sasakian, Kenmotsu, and cosymplectic manifolds, in this article we investigate the harmonic maps on  $C_{12}$ -manifolds which are non-normal almost contact metric manifolds. Finally, some examples are presented.*

**Keywords:** *Harmonic maps, Trans-Sasakian manifolds,  $C_{12}$ -manifolds, Kählerian manifolds.*

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## 1 Introduction

In [8], D. Chinea and C. Gonzalez have obtained a classification of the almost contact metric manifolds by studying the spaces that possess the same symmetries as the covariant derivative of the fundamental 2-form. This space is decomposed into twelve irreducible components  $C_1, \dots, C_{12}$ .

Most of the research related to almost contact metric structures is concerned with the normal structures, in particular Sasakian, Kenmotsu, cosymplectic, and trans-Sasakian manifolds. This conclusion is clearly embodied in the field of harmonic maps on contact metric manifolds studied by Inaus and Pastore [13] (see, for example, [10, 11, 13, 14, 16]).

Recently, several works have appeared related to an important class of almost contact metric manifolds called  $C_{12}$ -manifolds, which are not normal. We cite, for example, [3, 5, 6, 7].

In this paper, we have discussed harmonic maps on  $C_{12}$ -manifolds and also discussed the non-existence of such maps. In Section 2, we recall some basic definitions and basic formulas, which we will use later. In Section 3, first we give the necessary and sufficient conditions for a  $\varphi$ -holomorphic map  $f : M_1 \rightarrow M_2$  between two  $C_{12}$ -manifolds to be harmonic. After that, we show that for any  $\varphi$ -holomorphic map  $f : M_1 \rightarrow M_2$  from  $C_{12}$ -manifold to the trans-Sasakian manifold, such that  $\xi_2 \in (Im df)^\perp$ . Then  $f$  can not be harmonic, provided  $\beta_2 \neq 0$ .

It is also shown that if  $F : M \rightarrow N$  is a smooth map between the  $C_{12}$ -manifold and the Kähler manifold, then  $F$  is harmonic if and only if  $\psi \in Ker dF$ . Finally, we prove that if  $F : N \rightarrow M$  is a smooth map between the Kähler manifold and the  $C_{12}$ -manifold, then  $F$  is harmonic.

## 2 Preliminaries

### 2.1 Trans-Sasakian manifolds

An odd-dimensional Riemannian manifold  $(M^{2n+1}, g)$  is said to be an almost contact metric manifold if there exist on  $M$  a  $(1, 1)$ -tensor field  $\varphi$ , a vector field  $\xi$  (called the structure vector field) and a 1-form  $\eta$  such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \quad (1)$$

for any vector fields  $X, Y$  on  $M$ . In particular, in an almost contact metric manifold we also have  $\varphi\xi = 0$  and  $\eta \circ \varphi = 0$ .

The fundamental 2-form  $\phi$  is defined by  $\phi(X, Y) = g(X, \varphi Y)$ . It is known that the almost contact structure  $(\varphi, \xi, \eta)$  is said to be normal if and only if

$$N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0, \quad (2)$$

for any  $X, Y$  on  $M$ , where  $N_\varphi$  denotes the Nijenhuis torsion of  $\varphi$ , given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \quad (3)$$

Some special cases are worthy of attention. A contact metric structure is an almost contact metric structure with  $d\eta = \phi$  and a Sasakian manifold is a normal contact metric manifold, equivalently, if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

where  $\nabla$  denotes the Levi-Civita connection of  $g$  throughout this paper.

An almost contact metric structure is said to be almost cosymplectic if both  $\eta$  and  $\phi$  are closed. If in addition the structure is normal, the structure is said to be cosymplectic, equivalently,  $(\nabla_X \varphi)Y = 0$ .

An almost contact metric manifold is an almost Kenmotsu manifold if  $d\eta = 0$  and  $d\phi = 2\eta \wedge \phi$ . Moreover, if the structure  $(\varphi, \xi, \eta)$  is normal, the manifold is said to be Kenmotsu. This manifold, can be characterized through their Levi-Civita connection, by requiring

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

In [17], the author proves that  $(\varphi, \xi, \eta, g)$  is trans-Sasakian structure of type  $(\alpha, \beta)$  if and only if it is normal and

$$d\eta = \alpha\phi, \quad d\phi = 2\beta\eta \wedge \phi, \quad (4)$$

where  $\alpha = \frac{1}{2n}\delta\phi(\xi)$ ,  $\beta = \frac{1}{2n}div\xi$  and  $\delta$  is the codifferential of  $g$ .

It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X). \quad (5)$$

From this formula one easily obtains

$$\nabla_X \xi = -\alpha\varphi X - \beta\varphi^2 X, \quad (6)$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) + \beta g(\varphi X, \varphi Y). \quad (7)$$

It is clear that a trans-Sasakian manifold of type  $(1, 0)$  is a Sasakian manifold and a trans-Sasakian manifold of type  $(0, 1)$  is a Kenmotsu manifold. A trans-Sasakian manifold of type  $(0, 0)$  is called a cosymplectic manifold.

## 2.2 $C_{12}$ -manifolds

In the classification of Chinea and Gonzalez [8] of almost contact metric manifolds there is a class  $C_{12}$ -manifolds which can be integrable but never normal. In this classification,  $C_{12}$ -manifolds are defined by

$$(\nabla_X \phi)(Y, Z) = \eta(X)\eta(Z)(\nabla_\xi \eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi \eta)\varphi Z. \quad (8)$$

In [6] and [7], The  $(2n + 1)$ -dimensional  $C_{12}$ -manifolds is characterized by:

$$(\nabla_X \varphi)Y = \eta(X)(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi), \quad (9)$$

for all  $X$  and  $Y$  vector fields on  $M$  where  $\omega = -\nabla_\xi \eta$  is a differential 1-form orthogonal to  $\eta$  and  $\psi = -\nabla_\xi \xi$  is a vector field such that  $\omega(X) = g(X, \psi)$ . Firstly, notice that the vector field  $\psi$  is perpendicular to  $\xi$  because  $g(\psi, \xi) = -g(\nabla_\xi \xi, \xi) = 0$ .

Therefore, we present here the following definition:

**Definition 2.1** [6] Let  $(M^{2n+1}, \varphi, \xi, \eta, g)$  be an almost contact manifold.  $M$  is called almost  $C_{12}$ -manifold if there exists a one-form  $\omega$  which satisfies

$$d\eta = \omega \wedge \eta \quad \text{and} \quad d\phi = 0.$$

In addition, if  $N_\varphi = 0$  we say that  $M$  is a  $C_{12}$ -manifold and we denote it by  $(M, \varphi, \xi, \psi, \eta, \omega, g)$ .

In (9), putting  $Y = \xi$  one can get

$$\nabla_X \xi = -\eta(X)\psi \quad \text{and} \quad (\nabla_X \eta)Y = -\eta(X)\omega(Y). \quad (10)$$

For the three-dimensional, the following theorem has been proven

**Theorem 2.2** [4] Let  $(M^3, \varphi, \xi, \eta, g)$  be a 3-dimensional almost contact metric manifold.  $M$  is  $C_{12}$ -manifold if and only if

$$\nabla_X \xi = -\eta(X)\psi, \quad (11)$$

for all vector field on  $M$ , or equivalently,

$$\nabla_{\varphi X} \xi = 0. \quad (12)$$

**Example 2.1** We denote the Cartesian coordinates in a 3-dimensional Euclidean space  $\mathbb{R}^3$  by  $(x_1, x_2, x_3)$  and define a symmetric tensor field  $g$  by

$$g = \begin{pmatrix} \rho(x_1, x_2, x_3)^2 & 0 & 0 \\ 0 & \tau(x_1, x_2, x_3)^2 & 0 \\ 0 & 0 & \sigma(x_1, x_2, x_3)^2 \end{pmatrix},$$

where  $\rho$ ,  $\tau$  and  $\sigma$  are functions on  $\mathbb{R}^3$  and  $\rho\tau\sigma \neq 0$  everywhere. Further, we define an almost contact metric  $(\varphi, \xi, \eta)$  on  $\mathbb{R}^3$  by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \frac{1}{\rho} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \eta = (\rho, 0, 0).$$

Notice that  $d\eta = \rho_2 dx_2 \wedge dx_1 + \rho_3 dx_3 \wedge dx_1 = \omega \wedge \eta$  with  $\omega = \frac{\rho_2}{\rho} dx_2 + \frac{\rho_3}{\rho} dx_3$ , where  $\rho_i = \frac{\partial \rho}{\partial x_i}$  (The index always indicates the derivative). Therefore

$$\psi = \frac{\rho_2}{\tau\rho} e_1 + \frac{\rho_3}{\sigma\rho} e_2.$$

We give the following orthonormal basis

$$\xi = \frac{1}{\rho} \frac{\partial}{\partial x_1}, \quad e_1 = \frac{1}{\tau} \frac{\partial}{\partial x_2}, \quad e_2 = \frac{1}{\sigma} \frac{\partial}{\partial x_3}.$$

So, the components of the Levi-Civita connection corresponding to  $g$  are written

$$\begin{aligned}\nabla_{\xi}\xi &= -\frac{\rho_2}{\tau\rho}e_1 - \frac{\rho_3}{\rho\sigma}e_2, & \nabla_{\xi}e_1 &= \frac{\rho_2}{\tau\rho}\xi, & \nabla_{\xi}e_2 &= \frac{\rho_3}{\rho\sigma}\xi, \\ \nabla_{e_1}\xi &= \frac{\tau_1}{\tau\rho}e_1, & \nabla_{e_1}e_1 &= -\frac{\tau_1}{\tau\rho}\xi - \frac{\tau_3}{\tau\sigma}e_2, & \nabla_{e_1}e_2 &= \frac{\tau_3}{\tau\sigma}e_1, \\ \nabla_{e_2}\xi &= \frac{\sigma_1}{\rho\sigma}e_2, & \nabla_{e_2}e_1 &= \frac{\sigma_2}{\tau\rho}e_2, & \nabla_{e_2}e_2 &= -\frac{\sigma_1}{\rho\sigma}\xi - \frac{\sigma_2}{\tau\sigma}e_1.\end{aligned}$$

Using Theorem 2.2, one can check that  $(\mathbb{R}^3, \varphi, \xi, \eta, g)$  is a 3-parameter family of  $C_{12}$ -manifolds if and only if

$$\nabla_{e_i}\xi = -\eta(e_i)\psi = -\eta(e_i)\left(\frac{\rho_2}{\tau\rho}e_1 + \frac{\rho_3}{\sigma\rho}e_2\right),$$

where  $i \in \{0, 1, 2\}$  with  $e_0 = \xi$ , i.e.

$$\nabla_{\xi}\xi = -\frac{\rho_2}{\tau\rho}e_1 - \frac{\rho_3}{\rho\sigma}e_2, \quad \nabla_{e_1}\xi = \nabla_{e_2}\xi = 0.$$

From the above components of the Levi-Civita connection, we get

$$\tau_1 = \sigma_1 = 0.$$

### 2.3 Kählerian manifolds

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold  $(N, J, h)$  we thus have

$$J^2 = -1, \quad h(JX, JY) = h(X, Y), \quad (13)$$

for all  $X$  and  $Y$  vector fields on  $N$ . An almost complex structure  $J$  is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor  $N_J$  vanishes, with

$$N_J(X, Y) = J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y]. \quad (14)$$

For an almost Hermitian manifold  $(N, J, h)$ , we define the fundamental Kähler form  $\Omega$  as:

$$\Omega(X, Y) = h(X, JY). \quad (15)$$

$(N, J, h)$  is then called almost Kähler if  $\Omega$  is closed i.e.  $d\Omega = 0$  where  $d$  denotes the exterior derivative. It can be shown that this condition for  $(N, J, h)$  to be almost Kähler is equivalent to

$$h((\nabla_X J)Y, Z) + h((\nabla_Y J)Z, X) + h((\nabla_Z J)X, Y) = 0.$$

An almost Kähler manifold with integrable  $J$  is called a Kähler manifold, and thus is characterized by the conditions:  $d\Omega = 0$  and  $N_J = 0$ . One can prove that these both conditions combined are equivalent with the single condition

$$\nabla J = 0. \quad (16)$$

For more background on almost complex structure manifolds, we recommend the reference [18].

## 2.4 $\varphi$ -Holomorphic and harmonic maps

A smooth map  $f : M_1 \rightarrow M_2$  between two almost contact metric manifolds  $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$  with  $i \in \{1, 2\}$  is called to be a  $\varphi$ -holomorphic map if

$$df \circ \varphi_1 = \varphi_2 \circ df, \quad (17)$$

where  $df$  is the differential of  $f$ .

A smooth map  $F : M \rightarrow N$  between an almost contact metric manifold  $(M, \varphi, \xi, \eta, g)$  and an almost Hermitian manifold  $(N, J, h)$  is called to be a  $(\varphi, J)$ -holomorphic map if its differential intertwines the structures, that is

$$dF \circ \varphi = J \circ dF. \quad (18)$$

Let  $f : (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds. Then its differential  $df$  is a section of the bundle  $TM \otimes f^{-1}TN \rightarrow M$ . This bundle has a connection  $\tilde{\nabla}$  induced from the Levi-Civita connection  $\nabla^M$  of  $M$  and the pull back connection  $\bar{\nabla}$ . Then applying that connection to  $df$  gives the second fundamental form of  $f$

$$\tilde{\nabla} df(X, Y) = \bar{\nabla}_X^f df(Y) - df(\nabla_X^M Y), \quad (19)$$

for any  $X, Y \in \Gamma(TM)$ . It is easy to prove that a  $\tilde{\nabla} df$  is symmetric. The section  $\tau(f) \in \Gamma(f^{-1}(TN))$  defined by

$$\tau(f) = \operatorname{div} df = \sum_{i=1}^{2m+1} \tilde{\nabla} df(e_i, e_i), \quad (20)$$

where  $\{e_1, e_2, \dots, e_m\}$  is a local orthonormal frame on  $M$  is called the tension field of  $f$ .

We say that a map  $f : M \rightarrow N$  between two Riemannian manifolds  $M$  and  $N$  is harmonic if and only if [1, 9]

$$\tau(f) = 0. \quad (21)$$

## 3 Main results

### 3.1 Harmonic maps on $C_{12}$ -Manifolds

First, let us point out an important note regarding what was mentioned in some papers, such as [14, 15]. It is claimed in these two researches that for any  $\varphi$ -holomorphic map  $f$  between two Sasakian or trans-Sasakian manifolds we have

$$0 = df(\varphi_1 \xi_1) = \varphi_2 df(\xi_1), \quad (22)$$

which implies that

$$df(\xi_1) = 0. \quad (23)$$

But, if we apply  $\varphi_2$  in (22) we get

$$df(\xi_1) = \eta_2(df(\xi_1))\xi_2,$$

and this is not necessarily zero. Moreover, in [13] it is also proved that if  $f$  is a  $\varphi$ -holomorphic map between two contact metric manifolds then, there exist  $a \in \mathbb{R}_+^*$  such that  $df(\xi_1) = a\xi_2$ . Unfortunately, all results based on formula (23) in [14, 15] have failed so they require correction.

So, to confirm our remark, we propose a counterexample using the example in [15] itself.

**Example 3.1** Let  $(x, y, z)$  be Cartesian coordinates in  $\mathbb{R}^3$  and define a symmetric tensor field  $g$  by

$$g = \begin{pmatrix} y^2 + e^z & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix},$$

Further, we define an almost contact metric  $(\varphi, \xi, \eta)$  on  $\mathbb{R}^3$  by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \quad \xi = \frac{\partial}{\partial z}, \quad \eta = dz - ydx.$$

then,  $(\mathbb{R}^3, \varphi, \xi, \eta, g)$  is a trans-Sasakian manifold of type  $(-\frac{1}{2e^z}, \frac{1}{2})$ .

let  $f$  be a  $\varphi$ -holomorphic map from this trans-Sasakian manifold to itself defined by

$$f(x, y, z) = \left( c_1x - c_2y + c_3; c_2x + c_1y + c_4; c_1z - \frac{c_2}{2}y^2 + c_5 \right).$$

where  $c_i \in \mathbb{R}$ . Then

$$df = \begin{pmatrix} c_1 & -c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & -c_2y & c_1 \end{pmatrix}.$$

Easily, we see that  $df(\xi) = c_1\xi \neq 0$

### 3.2 Harmonic Maps between two $C_{12}$ -Manifolds

**Theorem 3.1** Let  $(M_i^{2n_i+1}, \varphi_i, \xi_i, \eta_i, g_i)$  with  $i \in \{1, 2\}$  be two  $C_{12}$ -manifolds and  $f : M_1 \rightarrow M_2$  be a Riemannian  $\varphi$ -holomorphic. Then,  $f$  is harmonic if and only if  $df(\psi_1) = \sigma^2\psi_2 - \sigma\xi_2(\sigma)\xi_2$ , where  $\sigma = g_2(df(\xi_1), \xi_2)$ .

**proof 3.2** Since  $f$  is  $\varphi$ -holomorphic map then, we have

$$0 = df(\varphi_1\xi_1) = \varphi_2df(\xi_1),$$

which implies that

$$df(\xi_1) = \sigma\xi_2 \quad \text{with} \quad \sigma = \eta_2(df(\xi_1)). \quad (24)$$

From (19), we have

$$\begin{aligned} \tilde{\nabla}df(\xi_1, \xi_1) &= \bar{\nabla}_{\xi_1}^f df(\xi_1) - df(\nabla_{\xi_1}^1 \xi_1) \\ &= \nabla_{df(\xi_1)}^2 df(\xi_1) - df(\nabla_{\xi_1}^1 \xi_1), \end{aligned}$$

with the help of (24) and (9), we get

$$\tilde{\nabla}df(\xi_1, \xi_1) = \sigma\xi_2(\sigma)\xi_2 - \sigma^2\psi_2 + df(\psi_1). \quad (25)$$

Also, from (19) and (9), we have

$$\begin{aligned} \tilde{\nabla}df(X, \varphi_1Y) &= \varphi_2\tilde{\nabla}df(X, Y) \\ &\quad + \eta_2(df(X))\left(\omega_2(\varphi_2df(Y))\xi_2 + \eta_2(df(Y))\varphi_2\psi_2\right), \end{aligned}$$

for any  $X, Y \in \Gamma(\mathcal{D}_1)$ . Since  $\tilde{\nabla}df$  is symmetric, we find

$$\begin{aligned} \tilde{\nabla}df(X, \varphi_1Y) - \tilde{\nabla}df(Y, \varphi_1X) &= \\ &\quad \left(\eta_2(df(X))\omega_2(\varphi_2df(Y)) - \eta_2(df(Y))\omega_2(\varphi_2df(X))\right)\xi_2. \end{aligned}$$

Now put  $Y = \varphi_1X$  in the above equation and doing the calculations we get

$$\tilde{\nabla}df(X, X) + \tilde{\nabla}df(\varphi_1X, \varphi_1X) = \eta_2(df(X))\omega_2(df(X))\xi_2. \quad (26)$$

Now the tension field of  $f$  is

$$\tau(f) = \sum_{i=1}^{n_1} \left( \tilde{\nabla}df(e_i, e_i) + \tilde{\nabla}df(\varphi_1e_i, \varphi_1e_i) \right) + \tilde{\nabla}df(\xi_1, \xi_1).$$

By virtue of (25) and (26) the above equations gives

$$\tau(f) = \sum_{i=1}^{n_1} \eta_2(df(e_i))\omega_2(df(e_i))\xi_2 + \sigma\xi_2(\sigma)\xi_2 - \sigma^2\psi_2 + df(\psi_1). \quad (27)$$

Since

$$\eta_2(df(e_i)) = -\eta_2(df(\varphi_1^2e_i)) = -\eta_2(\varphi_2df(\varphi_1e_i)) = 0,$$

thus

$$\tau(f) = \sigma\xi_2(\sigma)\xi_2 - \sigma^2\psi_2 + df(\psi_1),$$

### 3.3 Harmonic Maps between $C_{12}$ -Manifolds and trans-Sasakian ones

**Theorem 3.3** *Let  $(M_1^{2n_1+1}, \varphi_1, \xi_1, \eta_1, g_1)$  and  $(M_2^{2n_2+1}, \varphi_2, \xi_2, \eta_2, g_2)$  be a  $C_{12}$ -manifold and a trans-Sasakian manifold respectively. Suppose that  $f : M_1 \rightarrow M_2$  is a Riemannian  $\varphi$ -holomorphic such that  $\xi_2 \in (\text{Im } df)^\perp$ . Then  $f$  can not be harmonic, provided  $\beta_2 \neq 0$  and  $df(\psi_1) = -\sigma\xi_2(\sigma)\xi_2$ .*

**proof 3.4** *Firstly, one can check that*

$$\tilde{\nabla}df(\xi_1, \xi_1) = df(\psi_1) + \sigma\xi_2(\sigma)\xi_2.$$

*Depending on the above proof, from (19), (9) and (5) we have*

$$\begin{aligned} \tilde{\nabla}df(X, \varphi_1 Y) &= \varphi_2 \tilde{\nabla}df(X, Y) \\ &+ \alpha_2 \left( g_2(df(X), df(Y))\xi_2 - \eta_2(df(Y))df(X) \right) \\ &+ \beta_2 \left( g_2(\varphi_2 df(X), df(Y))\xi_2 - \eta_2(df(Y))\varphi_2 df(X) \right), \end{aligned}$$

*for any  $X, Y \in \Gamma(\mathcal{D}_1)$ . Using symmetry of  $\tilde{\nabla}df$ , we get*

$$\begin{aligned} \tilde{\nabla}df(X, \varphi_1 Y) - \tilde{\nabla}df(Y, \varphi_1 X) &= \\ \alpha_2 \left( \eta_2(df(X))df(Y) - \eta_2(df(Y))df(X) \right) \\ &+ \beta_2 \left( 2g_2(\varphi_2 df(X), df(Y))\xi_2 + \eta_2(df(X))\varphi_2 df(Y) - \eta_2(df(Y))\varphi_2 df(X) \right), \end{aligned}$$

*In the above equation, putting  $Y = \varphi_1 X$  and doing the calculations we obtain*

$$\begin{aligned} \tilde{\nabla}df(X, X) + \tilde{\nabla}df(\varphi_1 X, \varphi_1 X) &= -\alpha_2 \eta_2(df(X))\varphi_2 df(X) \\ &+ \beta_2 \left( -2g_2(df(X), df(X))\xi_2 + \eta_2(df(X))^2 \xi_2 + \eta_2(df(X))df(X) \right). \end{aligned} \quad (28)$$

*Now the tension field of  $f$  is given by*

$$\tau(f) = \sum_{i=1}^{n_1} \left( \tilde{\nabla}df(e_i, e_i) + \tilde{\nabla}df(\varphi_1 e_i, \varphi_1 e_i) \right) + \tilde{\nabla}df(\xi_1, \xi_1).$$

*By virtue of (28) the above equations gives*

$$\begin{aligned} \tau(f) &= df(\psi_1) + \sigma\xi_2(\sigma)\xi_2 - \alpha_2 \eta_2(df(e_i))\varphi_2 df(e_i) \\ &+ \beta_2 \left( -2g_2(df(e_i), df(e_i))\xi_2 + \eta_2(df(e_i))^2 \xi_2 + \eta_2(df(e_i))df(e_i) \right). \end{aligned} \quad (29)$$

*Since  $\eta_2(df(e_i)) = 0$ , we get*

$$\tau(f) = df(\psi_1) + \sigma\xi_2(\sigma)\xi_2 - 2\beta_2 g_2(df(e_i), df(e_i))\xi_2. \quad (30)$$

We can choose an adopted local orthonormal frame  $\{e_1, \dots, e_p, \varphi_1 e_1, \dots, \varphi_1 e_p, \xi_1\}$  for  $\ker df$  and  $\{e'_1, \dots, e'_q, \varphi_1 e'_1, \dots, \varphi_1 e'_q\}$  for  $(\ker df)^\perp$ , where  $2p + 2q = r + s = 2n_1$ . Thus, the tension field of  $f$  is given by

$$\tau(f) = df(\psi_1) + \sigma \xi_2(\sigma) \xi_2 - s \beta_2 \xi_2. \quad (31)$$

which confirms our Theorem.

**Corollary 3.5** *Let  $(M_1^{2n_1+1}, \varphi_1, \xi_1, \eta_1, g_1)$  and  $(M_2^{2n_2+1}, \varphi_2, \xi_2, \eta_2, g_2)$  be a  $C_{12}$ -manifold and an  $\alpha$ -Sasakian manifold respectively. Suppose that  $f : M_1 \rightarrow M_2$  is a Riemannian  $\varphi$ -holomorphic. Then  $f$  is harmonic if and only if  $df(\psi_1) = -\sigma \xi_2(\sigma) \xi_2$ .*

**Proposition 3.6** *Let  $(M_1^{2n_1+1}, \varphi_1, \xi_1, \eta_1, g_1)$  and  $(M_2^{2n_2+1}, \varphi_2, \xi_2, \eta_2, g_2)$  be an  $\alpha$ -Sasakian manifold and a  $C_{12}$ -manifold respectively and let  $f : M_1 \rightarrow M_2$  be a Riemannian  $\varphi$ -holomorphic. Then,  $f$  is harmonic.*

**proof 3.7** *One can adapt Theorems 3.1 and 3.3 .*

### 3.4 Harmonic Maps between $C_{12}$ -Manifolds and Kählerian manifolds

**Theorem 3.8** *Let  $(M^{2m+1}, \varphi, \xi, \eta, g)$  be a  $C_{12}$ -manifold and  $(N^{2n}, J, h)$  be a Kählerian manifold. Suppose that  $F : M \rightarrow N$  is a  $(\varphi, J)$ -holomorphic map. Then,  $F$  is harmonic if and only if  $\psi \in \text{Ker } dF$ .*

**proof 3.9** *To prove the harmonicity of  $F$ , it will be enough to prove that  $\tau(F) = 0$ . For a  $(\varphi, J)$ -holomorphic map we have the following formula for its tension field see [10, 12]*

$$J(\tau(F)) = dF(\text{div} \varphi) - \text{tr}_g \beta, \quad (32)$$

where  $\beta(X, Y) = (\tilde{\nabla}_X J)(dF(Y))$ .

Let  $\{e_1, \dots, e_m, \varphi e_i, \dots, \varphi e_m, \xi\}$  be an orthonormal  $\varphi$ -adapted local frame on  $TM$ . Then

$$\text{div} \varphi = \sum_{i=1}^m \left( (\nabla_{e_i} \varphi) e_i + (\nabla_{\varphi e_i} \varphi) \varphi e_i \right) + (\nabla_{\xi} \varphi) \xi. \quad (33)$$

From (9) we have

$$(\nabla_{e_i} \varphi) e_i = (\nabla_{\varphi e_i} \varphi) \varphi e_i = 0 \quad \text{and} \quad (\nabla_{\xi} \varphi) \xi = \varphi \psi.$$

Thus

$$\text{div} \varphi = \varphi \psi. \quad (34)$$

On the other hand as  $F$  is  $(\varphi, J)$ -holomorphic we have

$$J(dF(\xi)) = (dF(\varphi\xi)) = 0,$$

and thus  $dF(\xi) = 0$ . Now as  $N$  is a Kähler manifold we obtain

$$\begin{aligned} tr_g \beta &= \sum_{i=1}^m \left( (\tilde{\nabla}_{e_i} J) dF(e_i) + (\tilde{\nabla}_{\varphi e_i} J) dF(\varphi e_i) \right) + (\tilde{\nabla}_{\xi} J) dF(\xi) \\ &= \sum_{i=1}^m \left( (\nabla_{dF(e_i)}^N J) dF(e_i) + (\nabla_{JdF(e_i)}^N J) J dF(e_i) \right) = 0. \end{aligned}$$

Finally by using Formula (32), we obtain

$$J(\tau(F)) = dF(\varphi\psi),$$

i.e.

$$\tau(F) = dF(\psi),$$

which confirms our Theorem.

**Theorem 3.10** Any  $(J, \varphi)$ -holomorphic mapping  $F$  from a Kählerian manifold  $(N^{2n}, J, h)$  to a  $C_{12}$ -manifold  $(M^{2m+1}, \varphi, \xi, \eta, g)$  is a harmonic map.

**proof 3.11** For a  $(J, \varphi)$ -holomorphic map we have a similar formula as (30)

$$\varphi(\tau(F)) = dF(\operatorname{div} J) - tr_h \beta, \quad (35)$$

where  $\beta(X, Y) = (\tilde{\nabla}_X \varphi)(dF(Y))$ .

As  $N$  is a Kähler manifold we have

$$\operatorname{div} J = \sum_{i=1}^{2n} (\tilde{\nabla}_{e_i} J) e_i = 0.$$

where  $\{e_i\}_{1 \leq i \leq 2n}$  is an orthonormal local basis on  $TN$ . Now, using the formula (9) we obtain

$$\begin{aligned} tr_h \beta &= \sum_{i=1}^{2n} (\tilde{\nabla}_{e_i} \varphi) dF(e_i) \\ &= \sum_{i=1}^{2n} (\nabla_{dF(e_i)}^M \varphi) dF(e_i) \\ &= \sum_{i=1}^{2n} \eta(dF(e_i)) \left( \omega(\varphi dF(e_i)) \xi + \eta(dF(e_i)) \varphi \psi \right) = 0, \end{aligned}$$

because

$$\eta(dF(e_i)) = -\eta(dF(J^2e_i)) = -\eta(\varphi dF(Je_i)) = 0.$$

Thus

$$\varphi(\tau(F)) = 0,$$

that is

$$\tau(F) = \eta(\tau(F))\xi.$$

We have just obtained that  $F$  is harmonic if and only if  $g(\tau(F), \xi) = 0$ .

On the other hand

$$\begin{aligned} g(\tau(F), \xi) &= \sum_{i=1}^{2n} g\left(\tilde{\nabla}_{e_i}^F dF(e_i) - dF(\nabla_{e_i}^N e_i), \xi\right) \\ &= \sum_{i=1}^{2n} \left( g(\nabla_{dF(e_i)}^M dF(e_i), \xi) - g\left(dF(\nabla_{e_i}^N e_i), \xi\right) \right) \\ &= -\sum_{i=1}^{2n} g\left(\nabla_{dF(e_i)}^M (\varphi dF(Je_i)), \xi\right) + \sum_{i=1}^{2n} g\left(\varphi dF(J\nabla_{e_i}^N e_i), \xi\right). \end{aligned}$$

In the last equality we have used that  $N$  is Kähler and  $F$  is a  $(J, \varphi)$ -holomorphic map. Now the second term vanishes and we get

$$\begin{aligned} g(\tau(F), \xi) &= -\sum_{i=1}^{2n} g\left((\nabla_{dF(e_i)}^M \varphi)(dF(Je_i)), \xi\right) - \sum_{i=1}^{2n} g\left(\varphi \nabla_{dF(e_i)}^M (dF(Je_i)), \xi\right) \\ &= -\sum_{i=1}^{2n} g\left((\nabla_{dF(e_i)}^M \varphi)(dF(Je_i)), \xi\right), \end{aligned}$$

Using the formula (9) and the fact that  $\eta(dF(e_i)) = 0$  we get

$$g(\tau(F), \xi) = 0.$$

Therefore

$$\tau(F) = 0.$$

which completes the proof.

**Example 3.2** From Example 2.1, we have  $(\mathbb{R}^3, \varphi, \xi, \eta, g)$  such that

$$g = \begin{pmatrix} \rho(x, y, z)^2 & 0 & 0 \\ 0 & \mu(y, z)^2 & 0 \\ 0 & 0 & \sigma(y, z)^2 \end{pmatrix},$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \frac{1}{\rho} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \eta = (\rho, 0, 0),$$

is a  $C_{12}$ -manifold where  $\omega = \frac{\rho_2}{\rho} dy + \frac{\rho_3}{\rho} dz$ , and

$$\psi = \frac{\rho_2}{\mu^2 \rho} \frac{\partial}{\partial y} + \frac{\rho_3}{\sigma^2 \rho} \frac{\partial}{\partial z}.$$

We consider  $(\mathbb{R}^2, J, h)$  the standard Kählerian manifold where  $h = du^2 + dv^2$  and  $J\partial_u = \partial_v$ ,  $J\partial_v = -\partial_u$ .

The smooth map  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by

$$F(u, v) = (\alpha(x, y, z), \beta(x, y, z)).$$

$F$  is a  $(\varphi, J)$ -holomorphic map if and only if the matrix

$$\begin{pmatrix} \beta_1 & \frac{\sigma\beta_2 + \mu\alpha_3}{\sigma} & \frac{-\sigma\alpha_2 + \mu\beta_3}{\sigma} \\ -\alpha_1 & \frac{-\sigma\alpha_2 + \mu\beta_3}{\sigma} & \frac{-\sigma\beta_2 + \mu\alpha_3}{\mu} \end{pmatrix},$$

vanish. Using suitable software, the system of partial differential equations gives three solutions:

$$F = \text{const}, \quad F = (p(z), q(y)) \quad \text{with} \quad \sigma = -\frac{\mu p'}{q'},$$

$$F = (k(y, z), \beta(x, y, z)) \quad \text{with} \quad \sigma = -\frac{\mu\beta_3}{k_2}, \quad \beta_2 = -\frac{k_2 k_3}{\beta_3} \quad \text{and} \quad \beta_1 = 0.$$

We will continue to study this example using the second case i.e.

$$F = (p(z), q(y)) \quad \text{with} \quad \sigma = -\frac{\mu p'}{q'}.$$

That is

$$dF = \begin{pmatrix} 0 & 0 & p' \\ 0 & q' & 0 \end{pmatrix}.$$

Knowing that  $\nabla_{\partial_i}^{\mathbb{R}^2} \partial_j = 0$  where where  $\{\partial_1 = \frac{\partial}{\partial u}, \partial_2 = \frac{\partial}{\partial v}\}$  is a local orthonormal frame on  $\mathbb{R}^2$  and using the components of the Levi-Civita connection corresponding to  $g$  given in Example 2.1, one can check that

$$\tau(F) = \begin{pmatrix} 0 \\ \frac{q'^2 \rho_3}{\mu^2 p' \rho} \\ \frac{q' \rho_2}{\mu^2 \rho} \end{pmatrix} = dF(\psi).$$

Witch confirms Theorem 3.8.

To check the Theorem 3.10, we consider the smooth map  $H : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by

$$H(u, v) = (a(u, v), b(u, v), c(u, v)).$$

$H$  is a  $(J, \varphi)$ -holomorphic map if and only if the matrix

$$\begin{pmatrix} a_2 & -a_1 \\ \frac{\sigma c_1 + \mu c_2}{\mu} & \frac{\mu b_1 - \sigma c_2}{\mu} \\ \frac{\mu b_1 - \sigma c_2}{\mu} & -\frac{\sigma c_1 + \mu c_2}{\mu} \end{pmatrix},$$

vanish. So, the system of partial differential equations gives three solutions:

$$H = \text{const}, \quad H = (\lambda, ru + t, l(v)) \quad \text{with} \quad \sigma = \frac{r\mu}{l'},$$

$$H = (\lambda, s(v), mu + n) \quad \text{with} \quad \sigma = -\frac{\mu s'}{m}.$$

where  $\lambda, r, t, m, n \in \mathbb{R}$ . Using the second case i.e.

$$H = (\lambda, ru + t, l(v)) \quad \text{with} \quad \sigma = \frac{r\mu}{l'}.$$

That is

$$dH = \begin{pmatrix} 0 & 0 \\ r & 0 \\ 0 & l' \end{pmatrix}.$$

Now, with simple calculations we find  $\tau(H) = 0$ .

## 4 Open Problem

The question remains open about the stability of these results and their validity for  $\text{div}\psi = \text{const}$  or  $\omega$  is closed or for dimensions greater than three. In other words, what happens for the cases where  $\text{div}\psi \in \{0, 1\}$  or  $d\omega = 0$  or  $\dim M > 3$ ?

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